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ABSTRACT

Chapters present the relation of mathematics to and the use of mathematics in each of the following areas: social sciences, biology, religion, investment, agriculture, pharmacy, statistics, and physics. In addition there is a chapter about humanistic bearings of mathematics and a chapter explaining the aesthetic values of different polygonal forms. (LS)

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EDITOR'S PREFACE

This is the sixth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The first dealt with "A Survey of Progress in the Past Twenty-five Years," the second with "Curriculum Problems in Teaching Mathematics," the third with "Selected Topics in the Teaching of Mathematics," the fourth with "Significant Changes and Trends in the Teaching of Mathematics Throughout the World since 1910," and the fifth with "The Teaching of Geometry." Bound copies of all but the first of these Yearbooks can still be secured from the Bureau of Publications, Teachers College, Columbia University, New York, for $1.75 each.

These Yearbooks have been well received and have no doubt been the source of much help to teachers of mathematics and to others interested in the mathematics field. The success of these previous Yearbooks has convinced the National Council of the desirability and wisdom of continuing the series. The Sixth Yearbook is accordingly presented in the hope that it will be helpful not only to secondary teachers but to intelligent laymen as well.

The purpose of the book is to set forth as completely as possible in the space allotted the place of mathematics in modern life. Other chapters, treating of subjects like Mathematics and Engineering, for example, might have been included, but the place of mathematics in such fields is obvious, so it was decided best not to include them.

I wish to express my personal appreciation as well as that of the National Council to all who have contributed to the Yearbook or who in any way have helped to make it what it is.

W. D. Reeve.

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MATHEMATICS IN MODERN LIFE

X

THE APPLICATION OF MATHEMATICS TO THE SOCIAL SCIENCES *

By IRVING FISHER

Yale University, New Haven, Conn.

A Personal Tribute. It may not be amiss to precede what I have to say on Mathematics in the Social Sciences by a reminiscent statement of my personal impressions of J. Willard Gibbs, whose pupil I was forty years ago. J. Willard Gibbs towered, head and shoulders, above any other intellect with which I have come in contact. I had a keen realization of his greatness even in those formative years in Yale College and the Yale Graduate School. But this keen realization has grown even keener as the years have swept by, not only because of the increased evidence of the fundamental value of Gibbs' work in his own chosen field but also because in my own consciousness, after so many details have dropped from memory, there persists all the more clearly the strong impression which Gibbs' personality and teaching made upon me.

In saying this I do not think I can be accused of undue enthusiasm simply from the loyalty of a pupil to his teacher, especially in view of the statements of Lord Kelvin and others, which virtually rank Gibbs as the Sir Isaac Newton of America. Lord Kelvin said when visiting at Yale, a few years ago, that "by the year 2000 Yale would be best known to the world for having produced J. Willard Gibbs."

One of the most striking characterizations of Gibbs was recently made by Dr. John Johnston, now with the United States Steel Corporation, then Professor of Chemistry at Yale, in his address on Gibbs delivered at Yale University two years ago. He stated that no result of Gibbs' work had yet been overthrown, and that, in this respect, Gibbs seems to stand unique and supreme among the great scientists.

* Adapted by permission from *Bulletin of the American Mathematical Society*, April 1930. The seventh Josiah Willard Gibbs Lecture, read at Des Moines, December 31, 1929, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

The English physical chemist, Professor F. G. Donnan, has paid the following tribute to him:

Gibbs ranks with men like Newton, Lagrange, and Hamilton, who by the sheer force and power of their minds have produced those generalized statements of scientific law which mark epochs in the advance of exact knowledge. . . The work and inspiration of Gibbs have thus produced not only a great science but also an equally great practice. There is, to-day, no great chemical or metallurgical industry that does not depend, for the development and control of a great part of its operations, on an understanding and application of dynamic chemistry and the geometrical theory of heterogeneous equilibria.

Professor Ostwald said, in the preface to his German translation of Gibbs' thermodynamic papers in 1892:

The importance of the thermodynamic papers of Willard Gibbs can best be indicated by the fact that in them is contained, explicitly or implicitly, a large part of the discoveries which have since been made by various investigators in the domain of chemical and physical equilibrium and which have led to so notable a development in this field. . . . The contents of this work are to-day of immediate importance and by no means of merely historical value. For of the almost boundless wealth of results which it contains, or to which it points the way, only a small part has up to the present time 1892 been made fruitful.

Sir Joseph Larmor said of the work of Gibbs:

This monumental memoir *On the Equilibrium of Heterogeneous Substances* made a clean sweep of the subject, and workers in the modern experimental science of physical chemistry have returned to it again and again to find their empirical principles forecasted in the light of pure theory and to derive fresh inspiration for new departures.

We think no less of Gibbs' greatness because he himself showed so little consciousness of it. He must have realized the fundamental character of his work. But his pupils remarked his profound modesty and often commented on it. His chief delight was in truth-seeking for its own sake, and he was so intent on this search that he had no time even to think of emphasizing the originality or value of his own additions to the great vista of truth over which his mind swept. Doubtless he often did not know or greatly care where the work of others ceased and his own began. He did not always wade through the literature which preceded his own scientific papers. I remember hearing him say that when he wanted to verify another man's results he usually found it easier to work them out for himself than to follow the other man's own course of reasoning. This was said without any affectation, but simply in a jocular vein, as by one who would escape a difficult task by going his own way. But even though it may be difficult to disentangle what was original in Gibbs' work from what was anticipated by others, surely no competent person doubts to-day that he founded a new era in physics and chemistry.

Because of his retiring disposition and the theoretical nature of his work, he was, during his lifetime, almost unknown except among a few special students. The majority of students at Yale, in my day, did not know of his existence, much less of his greatness. And there were far fewer people in America who could appreciate what he was doing than there were in Europe.

His work was more promptly recognized in Germany. When I studied in Berlin in 1893, and was asked under whom I had studied in America, I enumerated the mathematicians at Yale. To my mortification, not one of the names was known to those Berlin professors, until I mentioned Gibbs, whereupon they were loud in his praises. "*Geebs, Geebs, jawohl, ausgezeichnet!*"

Even to-day, Gibbs illustrates the rule that a prophet is not without honor save in his own country. As Professor Johnston noted, the fiftieth anniversary of the publication of the first part of Gibbs' great work *On the Equilibrium of Heterogeneous Substances* was signalized in Holland by the publication of a Gibbs number of their chemical journal. This contained contributions from Dutch authorities, as well as from French, German, Canadian, Norwegian, and English authorities, but not one contribution from an American!

It is true, however, that at Yale we have finally established a Gibbs fund for a lectureship to be filled by visiting professors and that a new complete edition of his works has been issued by Longmans, Green and Co. It is also planned to issue two volumes of commentations on Gibbs' work to make its chief results more accessible to the general scientist. I am proud to have played a part in these undertakings. May I take this opportunity to say that I am also proud to have been included among the J. Willard Gibbs lecturers? And may I congratulate the American Mathematical Society on being the organization to found this lectureship, although Gibbs was professedly not so much a mathematician as a physicist. The only other Gibbs lectureship seems to be that at the Mellon Institute in Pittsburgh. The Chicago Section of the

American Chemical Society awards a "J. Willard Gibbs medal" annually, the recipient of which makes an address.

Gibbs' Contributions to Science. Presumably Gibbs' greatest contribution to science was his application of the laws of thermodynamics to chemistry. He made this almost a deductive science. Professor Bumstead said of his work:

> To an unusual extent, among the sciences which appeal to experiment, it can be, and has been, cast in a deductive form. Sir Isaac Newton said that "it is the glory of geometry that from a few principles it is able to produce so many things." Thermodynamics shares in this kind of glory; it has only two fundamental principles, of which the first is a statement of the conservation of energy as applied to heat, and the second states the fact (so deeply founded in general experience that it seems almost axiomatic) that heat will not, of itself, flow from a body at a lower temperature to one of a higher temperature. From these two simple principles, by an almost Euclidean method, a surprising number of facts and relations between work and heat, and various properties of bodies were deduced about the middle of the last century.

J. Willard Gibbs was certainly a master at producing many deductions from a few general principles. And it was just because of the generality of the principles from which he always insisted on starting that he succeeded in reaching such a wealth of conclusions.

In fact, it has always seemed to me that Gibbs' chief intellectual characteristic consisted in his tendency to make his reasoning as general as possible, to get the maximum of results from the minimum of hypotheses. I shall never forget, and have often quoted, an aphorism used by Gibbs, whether or not original with him, to the effect that "the whole is simpler than its parts." For instance, when he had a problem involving coördinates he preferred to employ an *indeterminate* origin, maintaining that his results were thereby rendered simpler and easier than if he took the origin at some apparently more convenient but special point in relation to the crystal or other conformation which he was discussing. When the origin is indeterminate it automatically effaces itself from all the general relations deduced.

Many, if not most, other investigators instinctively seek to solve special cases before attempting to solve the general case. Sometimes they pay a big penalty in needless experimentation. I remember Professor Bumstead, my fellow student at Yale, recounting with relish a conversation that Gibbs was reputed to have had with a youthful investigator who had made a laborious experimental study of certain relationships and who was, with pardonable pride telling Gibbs of his conclusion. After listening attentively Professor Gibbs quite naturally and unaffectedly closed his eyes, thought a moment, and said, "Yes, that would be true," seeing at once that the special result which this young investigator had reached was a necessary corollary of Gibbs' own more general results. For him, it required no experimental verification. The young man's work had, from Gibbs' viewpoint, been almost as much wasted as if it had been spent in a laborious set of measurements of right-angled triangles on the basis of which measurements he should announce as a new and experimental discovery that the square of the hypotenuse is equal to the sum of the squares of the other two sides. It is worth noting that, though Gibbs did his work in the Sloane Physics Laboratory, he never, as far as I know at least, performed a single experiment. His life work, stupendous as it was, and based as it was on concrete fact, consisted exclusively in new deductions from old results, the full significance of which no one else had been able to derive.

In his effort to represent physical relations by geometrical models and to portray the theory of electricity and magnetism by geometrical methods, Gibbs encountered the need of a new vector analysis to replace the awkward analysis by Cartesian coördinates, requiring, as that does, three times as many equations to write and manipulate as does vector analysis, not to say diverting attention from the lines and surfaces actually involved to their projections on three arbitrary axes.

To me the most interesting course I had with Gibbs was on vector analysis. He believed he had simplified the Hamilton system of quaternions, getting his cue from Grassmann's *Ausdehnungslehre*. But he was so conscious of his obligations to Grassmann that he was reluctant to publish his own system, apparently doubting whether it possessed enough originality to warrant publication. He therefore had privately printed a syllabus of his system, and this reprint was used by us in his class as a text. Only after many years did Professor E. B. Wilson construct a more elaborate textbook embodying Gibbs' principles of vector analysis.

It is a curious fact that, while Gibbs' work in thermodynamics was appreciated in Germany, his work in vector analysis was not. I remember the comment of Professor Schwartz at Berlin, when I undertook to defend Gibbs' vector analysis: "*Es ist zu willkürlich.*" The Germans felt in honor bound to restrict pure mathematics to mere elaboration of the proposition that one and one make two. Starting with this proposition, by successive additions or subtractions of unity we may, of course, by going forward, obtain all the positive integers; this is addition. Reversing, we obtain zero and the negative integers; this is subtraction. Then, by applying repeated addition or multiplication and repeated subtraction or division, repeated multiplication or involution, and repeated division or evolution, we arrive at fractions, surds, imaginary quantities, and finally the *complex variable* $x + yi$. But beyond this, by such processes no more general form of magnitude can possibly be derived; for, if we operate on a complex variable by addition, subtraction, multiplication, division, involution or evolution, under the recognized rules of algebra, we obtain simply other complex variables and nothing else whatsoever. Only by changing, or as the German critics would say, by violating, the fundamental rules of algebra faithfully followed in the above processes, such as the rule that $a \times b$ is equal to $b \times a$, is it possible to enter into any other realm of mathematics than that of the complex variable.

When I reported these criticisms to Gibbs, his comment was that all depends on what your object is in making those sacrosanct rules for operating upon symbols. If the object is to interpret physical phenomena and if we find we can do better by having a rule that $a \times b$ is equal not to $b \times a$ but to minus $b \times a$, as in the multiplication of two vectors, then, he said, the criticisms of the Germans are beside the point.

The fact is that Gibbs, though a great mathematician, was not primarily interested in mathematics as such. His interest lay in its applications to reality—in the substance rather than the form. All his contributions to pure mathematics were sought and found not as mere proliferations of formal and abstract logic but as by-products of his work in interpreting the facts of the physical universe.

The far-reaching effects of Gibbs' work apply not only to inorganic physics and chemistry but also to the organic world. One of the most elaborate reviews of Gibbs and his relation to modern science is by Lieutenant Colonel Fielding H. Garrison, M.D., Assistant to the Librarian of the Army Medical Library, Washington, D. C., in which he shows, among other things, the application of Gibbs' work to the equilibrium of heterogeneous substances in general physiology.

Despite Gibbs' retiring disposition and his disinclination for general society he was most cordial in his personal contact with colleagues and students and never seemed to lack time to give to anyone who chose to discuss the subject in which he was so deeply interested. He made on all a deep impression of kindliness. I well remember the remark of Percey Smith, now Professor of Mathematics at Yale, who was a fellow student of Gibbs with me, as we walked out of one of Gibbs' lectures, "What a *gentle* man he is!"

He enjoyed a joke, often laughed and excited laughter. He took pleasure in applying his mathematics in simple ways. One of his minor but fascinating papers before the Yale Mathematical Club was on "The Paces of a Horse," the writing of which was doubtless suggested by watching a horse which he had just purchased. Probably no one else ever put a horse through his paces as scientifically or amusingly as Gibbs did in that paper.

Gibbs himself never contributed to the social sciences. Apparently I am the only one of his pupils who, after first doing some teaching in mathematics and physics, became professedly an economist, although Professor E. B. Wilson, Gibbs' chief interpreter as to mathematics, has taken a lively interest in many lines of social science and statistics and was this year President of the American Statistical Association.

After several years of graduate study partly in mathematics under Gibbs and partly in economics under Sumner, the time came for me to write my doctor's thesis, and I selected as my subject *Mathematical Investigations in the Theory of Value and Prices.* Professor Gibbs showed a lively interest in this youthful work, and was especially interested in the fact that I had used geometric constructions and methods, including his own vector notation.

The late Professor Allyn Young of Harvard also made occasional use of vectors in his economic work. Another economic student and writer, a brilliant young Norwegian, Professor Ragnar Frisch, has latterly used the vector notation and says he could scarcely think without it. Professor Frisch will this year be Visiting Professor at Yale from the University of Oslo.

It is one of the handicaps of mathematics in the social sciences that there are so few who are trained in both lines for such study, and this particularly applies to any applications of Professor Gibbs' vector analysis. If vector analysis should become more widely understood and used by students in the social sciences, doubtless it would be more generally utilized, at least as a vehicle for thought.

Occasionally, and increasingly, the ideas and notations of the differential and integral calculus are applied by mathematical economists and statisticians. But, of course, most of the mathematics employed in the social sciences consists of simple algebra. There is a saying, which, by the way, was quoted by Gibbs in his address on Multiple Algebra, that "the human mind has never invented a labor-saving machine equal to algebra."

There are several fairly distinct branches of social science to which mathematics has been, or may be, applied. The chief of these may be distinguished as (1) pure economics, (2) the "smoothing" of statistical series, (3) correlation, and (4) probabilities, all of which overlap to some extent.

My own chief interest in social science, from a mathematical point of view, has been in the first of these four groups, pure theory.

1. Mathematics in Economic Theory. When I began my work in this field in 1891, mathematics in economic theory was looked at askance, despite the fact that many years before, as early as 1838, Cournot had written his brilliant *Researches into the Mathematical Principles of the Theory of Wealth.* This book later greatly stimulated Professor Edgeworth of Oxford and Professor Marshall of Cambridge, and to-day is ranked among the economic classics. The same may be said of Jevons' *Theory of Political Economy,* published in 1871. But in 1891, when my own economic studies began, even the work of Cournot was almost unknown to economists, and that of Jevons was little used. If one will turn the pages of the main economic literature of 1891 and earlier, he will find practically no formulas and no diagrams. But Walras and Pareto in Switzerland and Pantaleoni and Baroni in Italy, Edgeworth and Marshall in England, Westergaard and Wicksell in Scandinavia, and a few other students in other countries were using and defending the new method.

When, at the request of Professor Edgeworth, I read a slightly mathematical paper on the *Mechanics of Bimetallism* before the

Economic Section of the British Association for the Advancement of Science at Oxford in September, 1893. I well remember how, in the discussion of that and other mathematical papers, Professor Edgeworth was, as he expressed it, "damped" by the unfriendly criticism of these new methods by Professor Sidgwick and others.

But little by little, the usefulness of mathematics has come to be appreciated. Besides the older writers already mentioned and Auspitz and Lieben, whose work on price determination of 1889 was one of my first inspirations, there have gradually come into this field many younger writers, among whom may be mentioned Professor Henry L. Moore of Columbia University, Professor J. H. Rogers of the University of Missouri, Professor C. F. Roos of Cornell University, Professor Griffith C. Evans of Rice Institute, Professor Henry Schultz of the University of Chicago, Professor Harold Hotelling of Stanford University, W. A. Shewhart of the Bell Telephone Laboratories, Professors J. Maynard Keynes, A. C. Pigou, and Arthur L. Bowley of England, Professors Albert Aftalion and Jacques Rueff of France, Professor L. von Bortkiewicz and Dr. Otto Kühne of Germany, Professor Wl. Zawadski of Poland, Professor E. Slutsky of Russia, Professor Gustav Cassel of Sweden, Professor Ragnar Frisch of Norway, Dr. Willem L. Valk of Holland, Professors Corrado Gini and Luigi Amoroso of Italy.

And, besides the fact of such accessions to the ranks of the small band of professed mathematical economists, is the even more significant fact that economists in general have not only ceased decrying mathematics but are, in many cases, making some slight use of it themselves.

The late Professor Marshall of Cambridge University was one of the first to perceive what was happening. He said:

> A great change in the manner of thought has been brought about during the present generation by the general adoption of semi-mathematical language for expressing the relation between small increments of a commodity on the one hand, and on the other hand small increments in the aggregate price that will be paid for it: and by formally describing these small increments of price as measuring corresponding small increments of pleasure. The former, and by far the more important step was taken by Cournot (*Recherches sur les principes mathématiques de la théorie des richesses*, 1838); the latter by Dupuit (*De la mesure d'utilité des travaux publics*, in the Annales des Ponts et Chaussées, 1844), and by Gossen (*Entwickelung der Gesetze des menschlichen Verkehrs*, 1854). But their work was forgotten; part of it was done over again, developed and published almost simultaneously by Jevons and by Carl Menger in 1871, and by Walras a little later.

Jevons almost at once arrested public attention by his brilliant lucidity and interesting style. . . .

A training in mathematics is helpful by giving command over a marvellously terse and exact language for expressing clearly some general relations and some short processes of economic reasoning; which can indeed be expressed in ordinary language, but not with equal sharpness of outline. And, what is of far greater importance, experience in handling physical problems by mathematical methods gives a grasp, that cannot be obtained equally well in any other way, of the mutual interaction of economic changes. The direct application of mathematical reasoning to the discovery of economic truths has recently rendered great services in the hands of master mathematicians to the study of statistical averages and probabilities and in measuring the degree of consilience between correlated statistical tables.

Mathematics serves economic theory in supplying such fundamental concepts based on the differential calculus and also through the process of differentiation solves problems of maxima and minima, as in the simple process of determining formally what is the price that the traffic will bear in order to make profits a maximum.

The chief realm of economic theory to which mathematical analysis of this formal kind applies is that of supply and demand, the determination of prices, the theoretical effect of taxes or tariffs on prices. The results cannot always be reduced to figures but are often useful in terms of mere inequalities.

For instance, among the chief theorems shown mathematically by Cournot are the following:

That a tax on a monopolized article will always raise its price, but sometimes by more and sometimes by less than the tax itself.

That a tax on an article under unlimited competition always raises its price but by an amount less than the tax itself.

That a tax proportional to the net income of a producer will not affect the price of his product.

That fixed charges among costs of production do not affect price nor do taxes on fixed charges.

That opening up free trade in a competitive article between two previously independent markets may decrease the total product.

Among the most surprising paradoxes discovered by the mathematical method is one shown by Edgeworth, that if a monopolist sells two articles, say first and third class railway tickets, for which the demand is correlated, it may be possible to tax the third class tickets, at a fixed amount each, with the result that the monopolist not only pays the tax but lowers the prices of both kinds of tickets.

Familiarity with mathematics will save many confusions of thought. For instance, it is just as important in economics to distinguish between a *high* price and a *rising* price as it is in physics to distinguish between velocity and acceleration. *Rate of price change* has important effects, both theoretically and in practice, on the rate of interest and on the volume of business.

Theoretically the rate of interest ought to be higher during a period of rising prices, or depreciation of the dollar, by an amount equal to the rate of depreciation, and it ought to be lower during appreciation.

Practically, however, the rate of interest is slow of adjustment and what is more important, inadequate in adjustment. A mathematical statistical analysis of this slowness and inadequacy helps explain great business upheavals, as shown in my new book on *The Theory of Interest.* I may say here, parenthetically, though the case is somewhat different, that the recent crash in the stock market was, in large measure, the price paid for tardiness in raising the rate of interest, which should have been raised over a year before but was held down artificially.

Again, mathematics will save the student of economics and the student of accounting from the many confusions of double counting, especially in the intricate theory of income.

Another elementary, but important, use of mathematics in economics is in making sure that a problem is determinate by counting and matching the number of independent equations and the number of unknown quantities. A great deal of unnecessary misunderstanding has existed and still exists in economic science as to what determines the rate of interest or other magnitudes in economics. These misunderstandings would not exist if the contestants would take the trouble to express themselves mathematically. If we view the matter mathematically it soon becomes evident that one contestant has seen only one of the determining factors, and the other another, without either of them realizing that both are compatible and needed in a complete economic equilibrium. The concept of economic equilibrium in which many factors act and react on each other is one of the chief elementary contributions of mathematics to economic theory, and one stressed by Cournot, Walras, Marshall, Pareto, and Edgeworth.

Still another use of mathematics is in illustrating geometrically or analytically the fact that a price, or a *marginal utility*, is a function not simply of one but of many variables, the function being purely empirical and incapable of analytical or arithmetical expression. In fact, the economic world is a world of n dimensions.

Thus the marginal utility of bread to John Doe is a function of his quantity not only of bread consumed, but of butter, sugar, and numerous other variables.

I have myself tried to apply these and other mathematical ideas to the formal solution of the problem of prices of commodities, the rate of interest, the relation of capital to income, the purchasing power of money, and what Simon Newcomb, the astronomer-economist, called the equation of societary circulation, now called the equation of exchange (the volume of circulating medium multiplied by its velocity of circulation is equal to the price level multiplied by the volume of trade per unit of time).

Most of these and other applications of mathematics to economic theory consist in short chains of reasoning. Professor Marshall had the impression that only short chains of reasoning could ever be expected in mathematical economics. He said:

> It is obvious that there is no room in economics for long trains of deductive reasoning: no economist, not even Ricardo, attempted them. It may indeed appear at first sight that the contrary is suggested by the frequent use of mathematical formulas in economic studies. But on investigation it will be found that this suggestion is illusory, except perhaps when a pure mathematician uses economic hypotheses for the purpose of mathematical diversions; for then his concern is to show the potentialities of mathematical methods on the supposition that material appropriate to their use had been supplied by economic study. He takes no technical responsibility for the material and is often unaware how inadequate the material is to bear the strains of his powerful machinery.

But, as time goes on, there appear instances of somewhat longer trains of reasoning.

I may take an example from my own work. I have tried to show how it is possible to estimate numerically, through suitable mathematical equations, the velocity of the circulation of money. The formula for this was derived through a chain of mathematical reasoning requiring several links and embracing a considerable number of variables of which the chief are the volume of money in circulation, the annual flow of money into and out of banks, and the annual cash payments to labor. This problem, by the way, of evaluating the velocity of circulation of money had been pronounced insolvable and, without mathematical analysis, it might well be so considered. Incidentally, the calculations indicate that money in the United States circulates about twenty-five times a year. In other words, the average dollar stays in the same pocket about two weeks.

To turn to a different example, both Professor Ragnar Frisch and myself, by independent methods, both of them highly mathematical, have shown how, theoretically under certain simple hypotheses, to accomplish the statistical measurement of "marginal utility" or desirability, as a function of one's income. This would determine whether or not it is true that if one man has double the income of another his tax ought to be double, more than double, or less than double in order that he should make the same sacrifice. In other words, it would supply a mathematical criterion by which to judge the justice of a progressive income tax.

I say "would" rather than "does" simply because as yet the statistics available do not seem adequate for any accurate evaluation. But Professor Frisch and I are both hoping to pursue this study further. His and my preliminary results are not inconsistent. My own formula is derived by a chain of mathematical reasoning which results in expressing the ratio of the "marginal utility" of money for a person with a certain income to the "marginal utility" which he would have with a different income in terms of the following elements: those two incomes, the percentages which would be spent on food, rent, etc., under the two respective incomes and the index numbers of prices of food, rent, etc., relatively to another country, serving as a standard of comparison.

Mathematics also helps make clear the "dimensionality" of the magnitudes treated. Thus, the quantity of wheat, its price, and its value are three magnitudes as unlike in dimensionality as time, velocity, and distance. The rate of interest has the simplest dimensionality, being, like angular velocity, of dimension t^{-1}.

Mathematics helps us analyze time relationships in general, especially to avoid the old confusion between capital and income, the one relating to an instant of time, the other to a period of time.

Capital-income analysis is a development of the last two score years; but its roots go back for generations. Every good treatise on algebra includes the formulas for capitalizing annuities and bonds, the formulas underlying the bond tables used in every broker's office.

While the development of mathematical economics from the theoretical side has been steady and impressive since I was a young man, it has by no means been so rapid as the development of the three other branches to which I have referred.

2. Smoothing of Statistical Series. "Smoothing" statistical data, the fitting of formulas and curves to statistics, has, of course, been one of the aims of statisticians for many generations. In this way we have derived our mortality tables, the basis used by actuaries for calculating life insurance premiums.

I understand that the second J. Willard Gibbs lecture was by Robert Henderson on *Life Insurance as a Social Science and as a Mathematical Problem*. The importance of this branch of our subject does not need to be emphasized in an insurance center like Des Moines.

Actuarial science is, of course, a development of the formulas for capitalization or discount, particularly as applied to annuities, combined with the introduction of the probability element based on mortality statistics. It is essentially an analysis of interest and risk. It could be, and perhaps some day will be, applied to other economic problems besides life insurance, as soon as statistics are adequate for assessing risk numerically in other realms than human mortality. In fact, one of the crying needs of economic science is a reliable basis for evaluating risks.

Concurrently with actuarial science has developed a science of mathematics of mortality in relation to population, extending at least back to the days of William Farr, Superintendent of the Statistical Department of the Registrar General's Office of England half a century ago. To-day this science has been further developed by Knibbs of Australia, Lotka and Glover in the United States, and others.

Recently, with the development of statistics of industry, the art of curve fitting, by mathematical methods, has grown very rapidly, and examples of it will be found in many current issues of statistical journals. I am, myself, with a collaborator, Max Sasuly, working on a new method of curve fitting aimed to avoid the use of any preconceived formula but letting the statistical data themselves write their own formula, so to speak.

One important phase of curve fitting which links it closely with the study of economic theory is the statistical evaluation of supply and demand curves. Among those who have worked in this field are Professor Henry L. Moore of Columbia University, Professor Henry Schultz of Chicago University, Dr. Mordecai Ezekiel of the United States Department of Agriculture, Professors G. F. Warren, F. A. Pearson and C. F. Roos of Cornell University, and Professor Holbrook Working of Stanford University. Professor Schultz was apparently the first to work out the statistical determination of the effect of the tariff on the price of ar imported commodity—sugar.

3. Correlation. The third group of mathematical work in the social sciences, the development of correlation, is closely associated with the name of Karl Pearson of the University of London, who is still living. His "correlation coefficient" has become almost a standard procedure in economic statistics as well as in other sciences, including biology, in which he is primarily engaged. To-day many, if not most, economists, especially if they work in statistics, are making some use of such correlation coefficients. Through them they have been forced to adopt mathematical aids in spite of old traditions and prejudices.

Professor Warren M. Persons, formerly of Harvard, has worked out correlations with lag, showing the interrelations of various economic phenomena in such a way as to serve the purposes of forecasting business conditions. A more elaborate method of correlation has been worked out by Karl Karsten of New Haven, a private statistician, who has made tables of correlation between every pair of available series of economic statistics and has put these together by multiple correlation so as to predict any one of the series from all of the others which are found to serve toward that end. He is now issuing regularly a forecast of commodity prices, of the volume of business, of stock market trends, and of various other economic factors.

In the study of the so-called "business cycle" and forecasting future fluctuations, mathematical economists and statisticians have made increasing use of what is virtually differentiation or integration. Thus I have emphasized "price-change" as distinct from price, of which it is the differential quotient. Reciprocally, Mr. Karsten has applied the idea of "quadrature" to the relations of two statistical series where one is virtually derivable from the other by integration. This means if the curves are cyclical that they are related as are the curves of sines and cosines so that one curve is at zero while the other is at a maximum or minimum.

One development in this field in which I have been especially interested has been the study of the distribution of the lag. This appears to be a skew distribution, but nearly normal if time is plotted on a logarithmic scale.

As already indicated, risk is one of the fundamental elements in the mathematical analysis of actuarial science. It is also, in a different way, through frequency distribution, a fundamental element in correlation analysis. In fact, there have been more or less successful attempts by Karl Pearson to resolve a mortality curve into a sum of several frequency curves and by Arne Fisher to do the reverse, synthesize a set of frequency curves representing mortality from certain causes into the total mortality curve. It may also be pointed out that our second topic, curve fitting and smoothing, whether by least square methods or otherwise, is largely a study in probability.

4. Analysis of Probability. All this brings us to the fourth chief branch of our subject, the mathematical analysis of probability in general so far as this relates to social phenomena as embodied in statistics. This has been increasingly studied by many economists, especially the late F. Y. Edgeworth, editor of the *Economic Journal*. Also mathematical statisticians, such as G. Udny Yule, Arthur Bowley, R. A. Fisher, Sir William Beveridge, Truman L. Kelley, A. C. Whitaker, William L. Crum, Thiele, and others have done much constructive work in this field.

Professor Vilfredo Pareto tried to work out a formula for statistics of incomes in relation to the number of persons possessing incomes of various sizes, and the Pareto curve has become quite famous. It has been shown, however, particularly by Professor Macaulay of the National Bureau of Economic Research, that the Pareto curve is nothing but the "tail" of a probability curve, although Pareto himself had been loath to admit this. It is true that this particular sort of probability or distribution curve is not normal even if the abscissas are plotted on a logarithmic scale. It often happens in statistical series, especially where the frequency distribution lies between zero as one extreme and infinity as the other, that the frequency or probability curve while very skew on the arithmetical scale turns out to be nearly normal on the logarithmic scale.

I have, of course, by no means exhausted the list of applications of mathematics to economics, still less to the other social sciences. Many applications have been made which are not easily classified under the four heads I have noted, namely, pure theory, curve fitting, correlation, and probabilities.

Of these other and miscellaneous applications, the most important, at least in the field of economics and statistics, seems to be that of index numbers. The theory and practice of index numbers have had a special fascination for many of us because they occupy a tantalizing position on the borderline between *a priori* rational theory and empirical makeshift and because of the many pitfalls encountered in their study. It is closely related to the subject of probability. In my book on *The Making of Index Numbers*, I have tried to find the best formula for an index number out of several score available formulas.

It is also true, of course, that the last three divisions of our subject, curve fitting, correlation study, and probability, traverse all fields of knowledge. They apply not only to my own branch of the social sciences, economics, but to all others such as sociology, anthropology, and education, as well as to fields outside of social science such as psychology, biology, hygiene, and eugenics. In all of these we find a development of mathematical method. Each has its own special concepts, measures, and principles such as the cranial index of anthropology, the intelligence quotient of psychology and education, and the Mendelian principle in heredity; and these the mathematician may study in terms of averages, index numbers, correlations, deviations, frequency distributions, and otherwise. Just as the multiplication table is applicable in more than one field of knowledge, so mathematics in general is peculiar to none. Sooner or later every true science tends to become mathematical. The social sciences are simply a little later to be reached than astronomy, physics, and chemistry, while the biological sciences are later still.

Scientific method is one and the same, whether employed in such sciences as Gibbs developed, or in others. Mathematical notation is, as Gibbs said, simply a *language*. It is required for the best expression of scientific method when the relations to be expressed become too involved for ordinary language, which is less precise and complete. The outlook is bright for a healthy development of mathematics in the social sciences.

MATHEMATICS IN BIOLOGY *

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Introduction. What is mathematics? And what right has mathematics to obtrude itself upon the attention of biologists?

Since mathematics is a very old science, inextricably bound up in its historical development with logic and philosophy on the one hand and with astronomy on the other, it is perhaps impossible to give a concise definition which would satisfy the workers in all its fields, theoretical and applied. It is easier and more pertinent to our present purpose to indicate some of the characteristics of mathematics which make it an essential factor in the more advanced stages of the development of all other sciences. This may serve as a preliminary to an outline of the claims of mathematics to the attention of biologists, based on *a priori* considerations, on service in other natural sciences, and on the contribution which it has already made to the advancement of biology.

I. The Fundamental Characteristics of Mathematics as Related to the Physical and Biological Sciences

Premises and Conclusions. It is one of the characteristics of mathematics that, starting with certain axioms, postulates, or assumptions, it shows the way in which conclusions may be deduced from these premises. The mathematician does not necessarily claim absolute certainty for the physical validity of his conclusions, but he believes profoundly that it is possible to find groups of axioms—sets of a few propositions each—such that the propositions of each set are compatible and that the propositions of each set imply other

* The late Professor Harris was invited to write a chapter for this Yearbook on Mathematics in Biology. He gladly accepted the invitation and had started to prepare his material when he died suddenly after an operation for appendicitis. It seems fitting that we should have something from Professor Harris' pen. We are accordingly reprinting by permission this article adapted from *The Scientific Monthly* for August 1928. See also Harris, J. Arthur, "The Fundamental Mathematical Requirements of Biology." *American Mathematical Monthly*, 36: 179-198.—THE EDITOR.

propositions, which latter can be deduced from the former with certainty. The assumptions of pure mathematics need have no physical interpretations. They may indeed contradict any of our theories. They must not, however, contradict each other. Thus it is the business of the pure mathematician to discover systems of self-consistent and coherent propositions.

Mathematics is an exact and final science only in the sense that with the postulates definitely given the conclusion admits of no doubt. Thus the pure mathematician concerns himself not with the solution of particular problems in the natural sciences but with the principles which underlie the solution of problems in general.

The Natural Sciences. In the natural sciences, as contrasted with mathematics, the things which are given are measurements of natural phenomena rather than axioms or postulates. The scientist wishes to derive from these observational data generalizations which we call natural laws. It seems both reasonable and desirable that he should endeavor to employ in this task the rigorous self-consistent systems of reasoning developed by the mathematician.

The student of natural science need not necessarily concern himself with the criteria by which the mathematician assures himself of the validity of his formulas. He desires merely to be fully convinced of the usefulness of such formulas in the solution of the problems with which he himself has to deal. He should be sufficiently conversant with the fundamental assumptions underlying the equations which he proposes to use in his investigations, not to make the error of applying them in problems presenting wholly different sets of conditions from those for which they were developed.

The Biologist. The biologist in common with the student of the other natural sciences starts with a series of direct observations of phenomena. From these he wishes to derive a generalization, a theory, or a law which shall express the results of his experience in concise and mentally comprehensible terms. A theory deduced from a given set of observational data may be erroneous because the measurements were made under unsuitable conditions, or by inadequate methods, or because they are for some other reason unsuited for use as a basis of generalization. If the theories based on two or more sets of observations are inconsistent, the experimentalist refines the conditions under which his observations are made, increases the precision and number of his measurements, and reformulates his theories until he finds one which is in accord with the widest possible range of experience and which appeals to him as a reasonable description of the facts of nature. Thus, were the mathematician to criticize the worker in the natural sciences, he would not require him to give up observation and experimentation, but would only demand that the conclusions drawn from observation and measurement be logical conclusions.

II. The Claims of Mathematics on the Attention of Biologists

Claim of Pure Mathematics. The claim of pure mathematics to the attention of scholarly mankind is like that of art, in that it is grounded in the innate human love of beautiful things and in the innate human joy in originating them. It is creative. Its previous triumphs of achievement fill us with satisfying wonder; its pursuit is akin to that of the exploration of a great mountain system in the course of which the vantage of each peak scaled opens to view the prospect of higher serried peaks and vaster plateaus beyond.

The pursuit of pure mathematics in our day by the few who have the ability and the training is not to be justified by the immediate applicability of its results in the other sciences. Its present existence is justified by centuries of persistent appeal to human interest. In no generation in which we would deem life worth while has such interest in mathematics been lacking. Thus its claim to the attention of the scholar has been tested by a rigorous natural selection. The survival of the science is sufficient evidence for its value as a source of gratification to the active human mind. Furthermore, the process of natural selection has not merely shown the fitness of pure mathematics as such for survival. It has been active within mathematics itself. All that has not been found to be sound and consistent has been ruthlessly eliminated. Thus all that has long remained may properly appeal to workers in the other sciences as worth their consideration with reference to its possible value in their own special field.

But here we are not concerned with the justification of pure mathematics, but with an appeal for the wider use of mathematics by biologists as a means to the development of their own special field of creative scholarship.

Claims of Mathematics on the Working Biologist. Let us consider in outline the claims which mathematics has on the attention of the working biologist.

The first two may be stated in very general terms. They will, however, be developed in the discussion of special claims which is to follow.

The most general contribution of mathematics to the natural sciences is the affording of an exact and easily workable symbolism for the expression of ideas. The progress of science depends very largely upon the facility with which facts may be recorded and relationships between them considered. In their bearing upon this requirement of the natural sciences it is important to note that an essential characteristic of mathematical methods is that they economize thought. The notation of the mathematician affords the maximum precision, simplicity, and conciseness. The worker in natural science finds in mathematical literature a highly perfected symbolism which he may use without developing one of his own.

But while a convenient notation is the most general contribution of mathematics to the natural sciences, it is neither the only nor the most important one. In the natural sciences it is essential that accurate observations and exact measurements be interpreted by sound processes of reasoning. It seems logical to assume that the biologist may profit by the centuries of experience of the mathematician in the drawing of inevitable conclusions.

These claims are so general that we may properly turn to those based on the specific accomplishments of mathematics in the physical sciences and in biology itself in substantiation of our argument for its wider application in biological research.

The Claim of Service in Other Physical Sciences. The record of service of mathematics in the physical sciences is an outstanding claim on the attention of biologists.

In the past, mathematics has been an integral part of the sciences which we are accustomed to regard as the more highly developed—of all that is physical as distinguished from biological in the growth of our civilization. The most determined critic of the application of the mathematical method in biology dares not contemplate the consequences of a Maxwellian demon snatching from our scientific literature and from the minds of our chemists, physicists, engineers, and economists the mathematical formulas which underlie the routine of our daily life. In a few weeks long-distance communications would cease, the vehicles of transportation would be motionless, factories would close, and urban population would face starvation. As Professor A. Voss said in 1908, our entire present civilization, as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences.

If reasoning by analogy is ever justified, experience in the physical sciences would certainly seem to afford sufficient evidence of the necessity for the extensive introduction of this powerful tool of research into the biological sciences.

The argument that biologists should emulate the workers in the physical sciences is strengthened by the fact that biological phenomena are the most nearly infinitely complex of all natural phenomena. This is necessarily true because the internal structure and functioning of the organism and the effective environmental conditions under which it must live and reproduce comprehend a material fraction of the physical and chemical complexities of the universe. Before the more complicated biological phenomena can be grasped in any but the most circumscribed and superficial way by the human mind, they must either be analyzed and simplified by experimental control or expressed in the mentally intelligible terms of mathematical summaries or generalizations.

It may be urged that the method of dealing with large numbers of measurements is not that of the physicist or of the chemist who frequently works with minute samples under carefully controlled conditions.

The reasons for the differences in methods are two. First, the student of molecules has the advantage of working with less complex materials and under more readily controlled experimental conditions. Second, the physicist or chemist already has his molecules or ions massed and can investigate them and draw conclusions concerning their properties from his examination of the properties of his volume of gas or solution. The biologist must begin otherwise. He must collect and determine the characteristics of each individual of a large sample in order to express the characteristics of the whole population in mathematical terms.

When biologists have had the necessary preliminary training, they will realize that, for many of the phenomena with which they have to deal, the most easily comprehensible and the most useful method of description and analysis is the mathematical. In the past, biologists as a class have been in reality hostile to the introduction into their science of the methods which have proved their worth elsewhere. I know this to be true from long and bitter experience. Instead of being eager to place biology alongside of physics and chemistry in the ranks of the exact sciences, biologists have seemed not merely to excuse, but actually to take pride in the distinction which has been drawn between the so-called exact and the so-called descriptive sciences.

While the historical attitude of the biologist is not excusable, the fault has not been entirely his. With most men mathematics is like a well—the deeper they go in the less they see out and about. Mathematics may quite properly be an end in itself, but in biology it is strictly a means to an end. While mathematicians have in the past been eager to serve workers in the physical sciences, and while mathematics itself owes a large debt to these sciences, mathematicians have not for the most part felt it worth while to come to the assistance of biologists. Mathematicians have often asserted the need of mathematics in the biological sciences, but the claim has too often been made in an *ex cathedra* manner by those who, while perhaps qualified to speak of things mathematical, have been relatively little fitted to discuss the needs of biology. While biologists have been entirely too slow in recognizing the needs of their science for the mathematical tools, they have shown that practical good sense which characterizes those whose minds have contact with matter by refusing to flock to the mathematicians' standard until shown by concrete examples that the mathematical method has real applicability in biology. Thus the burden of proof has largely been thrown upon a few workers of greater vision, with the inevitable result that progress in the application of mathematics in biology has been slow.

Progress has been slow, but progress there has nevertheless been.

The Evolution of Biology and the Influence of Quantitative Methods. The natural sciences all had their beginnings in observation and speculation. Careful description of the observed phenomena then furnished a basis of interpretation by comparison. Experimentation, which requires not merely controlled conditions but measured consequences, followed observation and description. Finally quantitative measurement, calculation, and the formulation of mathematical laws have characterized the highest stage of scientific development.

These stages in the development of the natural sciences are, to be sure, neither wholly distinct in nature nor sharply separated in time. The methods of the later stages have in some instances been anticipated by investigators who were in advance of their contemporaries. It would be unfortunate indeed if men of science did not at all times avail themselves of whatever is best in the methods as well as in the results of those who preceded them. Notwithstanding the difficulty of delimiting the various horizons, as our geological friends might feel inclined to designate the deposits of scientific literature of these periods of differing dominant purposes, the sequence is in full accord with historical facts.

The old physicist who defined the biologist as "a man with scientific aspirations and inadequate mathematics" would find, if he looked over a fair sample of current biological literature, that not only has the space devoted to quantitative data increased enormously during the past few years, but that there is a steadily growing effort on the part of biologists to express in concise formulas the results of observation. Unfortunately, biology in most of its phases still lacks the quantitative data, and biologists in general want the training in mathematical analysis which is essential in exact science. Nevertheless the tendency of the times is unmistakable; the demand for quantitative work is more and more dominant in the biology of to-day.

The most forceful argument for the wider use of mathematics in biology is furnished by the service which mathematics has already rendered in the biological sciences. Let us consider this more specifically.

The Two Fronts of the Advance of Mathematics into Biology. Progress in science depends upon evolution of method as well as upon the accumulation of the data of observation, experimentation, and measurement. The progress which has been made in the development of biology as a quantitative science through the introduction of mathematical methods is in its present stage the resultant of various factors, which can be understood only when considered in their relation to the evolutionary history of science in general and of biology in particular.

This evolution of the natural sciences is admirably illustrated by the history of biology. Observations and speculations began with primitive man. If a desire to record what has been seen formed a part of the motives of those who bruised crude figures on the walls of caves, descriptions began with or before the period of written language. Some attempts at classification were made at a very early period in man's cultural development, but we are accustomed to think of the great era of description and classification as initiated in their modern form by the work of Linnaeus. This period was also one of detailed geographic exploration. Breadth of exploration doubtless tended to stimulate intensity of interest in description and classification. The activities of these decades resulted in the storing of great museums with carefully preserved and minutely described specimens of plants and animals, in the publication of elegant icones which are among the masterpieces of artistic book-making, comprehensive monographs of every large genus, encyclopedic summaries of phyla and kingdoms, and floras and faunas to the end of long vistas of library shelves.

Simultaneously with the latter decades of the period of description and classification of organisms, both living and fossil, began the development of anatomy and embryology, both macroscopic and microscopic. These latter were indefatigably pursued by an army of workers whose investigations were so comprehensive that the younger and more restless spirits began to fear that there would be no worlds left for them to conquer.

With such a wealth of descriptive materials at their disposal, it was inevitable that serious attempts at interpretation should be made. Speculation as to the observed phenomena was largely replaced by effort at interpretation based upon comparison. "It is descriptive but not comparative," was the criticism of a volume laid before the elder Agassiz. The dominance of the comparative method over a considerable period of the more recent history of biology is attested by the presence of the word *comparative* in the titles of a number of institutions and journals.

With taxonomy, comparative anatomy and embryology, histology, and cytology well outlined, biologists found themselves free to extend to other fields the methods which had heretofore been limited to physiology. Experimental morphology, experimental embryology, and experimental evolution are terms which illustrate the degree to which the experimental method has dominated biological investigation during the last few years.

a) The Influence of Physics and Chemistry. As soon as biology, in the course of its evolution, had passed the purely observational and descriptive stage and become an experimental science, it was natural that the attempt should be made to interpret biological phenomena in terms of the more highly developed sciences of physics and chemistry.

That many of the processes which occur in the living organism are chemical and physical in nature was necessarily admitted as soon as physiology could be called a science. The controversy between those who asserted that all biological phenomena are physical and chemical and those who maintained that living matter is in some essential way different from non-living matter has only served to stimulate investigations having to do with the physics and the chemistry of life processes.

The development of the field of physical and chemical physiology has been due not merely to its great theoretical interest but to its enormous practical importance in agriculture, in the industries, and in medicine. At present biophysics and biochemistry have attained the rank of independent sciences, commanding facilities and personnel greater than that available for the whole of biology, with the exception of taxonomy, a few years ago.

The intimate contact with the more precise sciences of physics and chemistry which has resulted from the rapid development of experimentation in biology during the past few years has done much to raise the standard of biological research.

Physics and chemistry are not merely sciences characterized by measurement rather than observation. They are sciences in which it has long been recognized that progress depends upon the exactness of the control of the conditions of experimentation, the precision of the measurements, and the adequacy of the mathematical description and analysis of the measurements which have been made. Here we have one of the two great lines of advance of the mathematical method into the biological sciences. Physics and chemistry are quantitative and, to a high degree, mathematical sciences. Biologists, if they will pursue their science along the lines of physics and chemistry, must take over the mathematical methods of expression and analysis characteristic of these sciences. There can be no reasonable doubt that in the future physics and chemistry will continue to influence biology, and even more profoundly than in the past. As the association of these sciences becomes more intimate, and as the biologist becomes essentially a chemist or a physicist working with living organisms, the mathematical mode of description and analysis which has been so fruitful in physics and chemistry will become increasingly significant in biology.

b) The Rise of Biometry. The penetration of the mathematical leaven into the biological lump through the medium of physics and chemistry has been so gradual and so little associated with the names of any individual workers that it has taken place without biologists as a class being acutely aware of the profound change in their science. The case is quite different with the second great line of advance of the mathematical methods into biology. This is directly traceable to the development, initiated by Francis Galton and strenuously carried forward by Karl Pearson, of mathematical formulas suitable for the analysis of the highly variable data of biological observation and measurement; and to the application of these methods to a wide range of biological and sociological problems by the biometric school.

While the biometric methods were developed primarily for the study of phenomena which are so complex that they cannot be grasped by the unaided human mind or which cannot be readily subjected to experimental control, they are now being advantageously applied to the results of experimentation. Biologists will doubtless some day realize that experimental results must receive mathematical treatment for their full interpretation.

For the present, there are many who stubbornly refuse to see.

We are sometimes told that the biometric constants are merely a useful means of expressing results. The idleness of such an assertion will be apparent from two simple illustrations.

All mankind has had the opportunity of observing the statures and other physical characteristics of husbands and wives. Yet it remained for Pearson and his group to show that there is a high degree of assortative mating in man. Why was this not perceived if the correlation coefficient only serves to express what we may learn otherwise?

If the suggestion be made that those individuals who observed human husbands and wives were for the most part scientifically untrained, the reply is evident. Students by the thousands in the biological laboratories of the world have observed conjugation in Paramecium, but it required the biometric investigation by Pearl, working under the influence of Pearson, to show that in the union there is a high degree of similarity in the size of the conjugants. Even after the relationship was clearly demonstrated biometrically, its validity was denied by at least two eminent zoölogists. If biometric methods are a useful means of expressing but not of obtaining results, why did not zoölogists long ago note the assortative conjugation demonstrated by Pearl, and arrive at the explanations afforded by the masterly studies in the same field by Jennings?

The answer is obvious in both cases. Unaided observation was incapable of dealing with the problems. They required for their solution the application of mathematical methods of analysis to series of measurements.

These are by no means unique or exceptional cases. Instances of the failure of biologists to observe important relationships, even with the materials or the data before their eyes, could easily be multiplied. Examples of the misinterpretation of materials or data equally open to observation could be readily adduced. The mental limitation implied is not peculiar to biologists. The inability to grasp the more complicated natural phenomena without symbolism is an inherent limitation of the human mind, fully recognized by psychologists. That a man should be unable to reason about highly complicated phenomena without the use of mathematical formulas is no more remarkable than that he should be unable to see chromosomes without the microscope.

Another criticism frequently heard is that the statistical methods can only locate problems—never solve them. The real solution, we are told, must in the end be biological, psychological, sociological, as the case may be. If this be true, it is the more important that the biologist, psychologist, and sociologist be themselves capable of using the mathematical methods, or at least of coöperating intelligently with those who can. But is the criticism really valid? The same stricture is equally applicable to all methods of research. After a group of phenomena have been described and analyzed as well as they can be by any means, other problems remain to be attacked by new refinements of method or of analysis.

The assertion is often made that the final results must depend upon the original measurements and not upon their mathematical treatment. A full discussion of this criticism would lead into several complexities, but it is sufficient to answer by a very simple illustration. The possibility of securing accuracy beyond the power of observation, or at least beyond the degree of refinement of the measurements adopted, may be easily tested by measuring a series of objects twice, once roughly and once with great accuracy. The statistical constants of these two series of measurements may then be calculated and compared. Unless there has been a consistent bias or personal equation on the part of the observer which tends to make all his measurements too high or too low, there will be a remarkably close agreement between the results of the constants calculated from the gross and from the refined series of measurements.

Finally, one of the most common criticisms of the biometric methods is that they are complex and difficult to use. We have been told seriously by biologists that they expect to adopt the biometric methods when they shall have been more simplified and hence made more suitable for practical use. But research does not tend to become simpler with the advance of science. Since biological phenomena are innately complex, there is no likelihood that the mathematical formulas required for their investigation will be simplified except in matters of practical technique. Criticism of the biometric methods on the ground of their difficulty is merely the glorification of the mental lassitude of the critic.

Let us turn from the answering of criticisms to things more constructive.

If science is to advance at the rate which we desire, another highly practical consideration cannot be neglected. Many biological phen mena cannot be subjected to experimental control. Thus while the proper study of mankind may be man, human individuals and their relatives cannot be investigated in the same manner as white rats and Drosophila. While man may be the most conspicuous illustration of an organism which cannot be studied in a broad way under controlled conditions, the example is not unique. In innumerable cases the statistical study of masses of data may not only properly, but must necessarily, replace controlled experimentation. I hope to show later that in such cases the experimental and the statistical method are in essence identical.

Even where refined experimentation is possible the biometric methods are particularly suited to reconnaissance work. In the search for the relationship between different variables the statistical analysis of large masses of comparatively rough data may indicate the place in which carefully controlled experiments may and should be made. Finally, after biological problems have been subjected to as close experimental control as possible, the results are generally so irregular as to make biometric analysis desirable.

Let us consider briefly, in review, the claims of the biometric methods to the attention of biologists.

First: The biometric notation makes possible the expression of the results of extensive experience in concise and mentally comprehensible terms.

This matter of the form of expression is one of far greater importance than might at first be realized. Rapidity of progress in any branch of science must depend very largely upon the facility with which the data and conclusions of a new investigation can be compared with those already on the library shelves. It is by the reoccurrence of like results that general theories are established. It is by the noting of inconsistencies and the circumstances under which they occur that indications of as yet unsuspected relationships are often seen.

There can be little doubt that the rapid advance of physics and chemistry has been due in no small degree to quantitative and standardized modes of expression.

If the physicist or chemist wants a solubility, melting point, or conductivity of any substance, he has merely to turn to volumes of constants to find whether it has been determined, and if constants are available, whether the recorded results accord with his own. An investigator has been able to draw upon a common fund of knowledge to a greater extent and with greater ease than in biology. Thus synthetic work has been facilitated.

In its bearing on the problem of the simplification of scientific literature, consider for a moment the state in which biology would be to-day had it not been for the Linnaean notation, by which species may be designated by a simple binomial instead of by a cumbersome description whenever it is mentioned. The value of this relatively succinct notation becomes especially apparent when we contemplate the vast harm which has been done to scientific research through the unwillingness or inability of taxonomists to maintain uniformity of nomenclature. Then in view of what has been accomplished by this relatively simple expedient, imagine the rapidity of advance which will be possible when a quantitative mode of expression permits the results of many fields of biological research to be summarized in annual volumes of standard constants. I, personally, am inclined to look upon the publication of Donaldson's volume on the rat, in which the experience of a whole institution of workers is summarized in quantitative terms, as a real milestone in the progress of biology.

Second: The biometric formulas provide a system of probable errors which safeguards the worker in the formulation of his conclusions. Biometricians have referred so freely to probable errors that critics have facetiously suggested that biometry is chiefly error. But frankly and candidly, if a given set of observations is insufficient to demonstrate a relationship, is it not better that the investigator discover the fact himself than that he should publish erroneous conclusions which must be corrected by subsequent research?

Third: The biometric methods not merely furnish a system of mentally comprehensible constants and concise equations, suitable for the description of complex phenomena, and a series of probable errors which safeguards the worker in drawing conclusions concerning these phenomena, but they make possible the investigation of relationships so intricate and so delicate that they are quite beyond the scope of unaided observation. Here the biometric methods have a potentiality for service analogous to that of the equipment of the modern observatory, which is capable of dealing with stellar phenomena that were beyond imagination a century ago, or to that of modern microscopic equipment and technique which have given rise to whole sciences of microcosms which were beyond the ken of Linnaeus. To argue that it is unnecessary to push on into the investigation of these more recondite relationships is as contrary to the spirit of science, as reactionary, as to argue that it were better to have stopped with Galileo instead of advancing to the refinements of modern astronomy through the development of instruments and mathematical theory.

Fourth: For many classes of problems the biometric formulas applied to large masses of data furnish the closest possible approximation to the experimental method of investigation.

The experimental method, as ideally applied, consists essentially in the simplification of conditions by rendering constant all but one. This one factor is then varied and its influence upon the organism is noted. In certain phases of statistical analysis an essentially identical method is followed, when we determine what is called the partial correlation between two variables for constant values of one, two, or more other variables.

For example, the basal metabolism of tall men is on the average greater than that of those of less stature. Heavy men also show a higher daily gaseous exchange than light ones. But taller men are on the average heavier men, and it seems quite possible that the larger basal food requirement of taller men is merely the resultant of the relationship between stature and body weight on the one hand and between weight and metabolism on the other. The biometrician solves such a problem statistically by determining the partial correlation between stature and metabolism for constant weight, i.e., with the influence of body weight eliminated. The experimentalist would have to attack the problem in exactly the same manner.

Illustrations might be given by the score of the analytical treatment of statistical data which gives results of essentially the same nature as those which are attained by the experimental method, often in cases in which strictly experimental technique cannot be readily applied.

Fifth: The biometric formulas furnish the best means as yet available for predicting the value of one variable from another, or from a series of others. This is due to the fact that it is possible to pass at once from measures of interdependence in terms of the universally comparable scale of correlation to regression equations showing the rate of change in terms of the actual working scale of any variable associated with another, or others, whose values are known.

The great theoretical importance of this feature of the biometric methods will be clearly realized when we remember that the test for the validity of a theory is its capacity for predicting the unknown.

The foregoing treatment in outline may have been disappointing to those who have expected argument by illustration of specific accomplishment. The method has been followed because the biological contributions which have already been made through the use of the biometric methods are now so large that no one man, even with unlimited space, can be expected to summarize them. This is true, notwithstanding the fact that the number of workers who have persistently stood by the biometric guns during the long and discouraging years of general indifference on the part of biologists can be counted on the fingers without using all the digits of the hands.

III. The More General Service of Mathematics in Biology

Certain Limitations and Services of Mathematics in Biology. It would be unfortunate to bring this paper to a close without emphasizing both certain limitations and certain wider services of mathematics in biology.

The biological universe is all but infinitely complex. It is not conceivable that all biological phenomena will be treated by mathematical methods. All that is necessary, however, is that the mathematical methods of research be so developed that they may be applicable to any biological problem.

Nor must there be misunderstanding concerning the desirability of an unbroken front comprising all the methods of research in the attack on the complex problems of the biological universe. In biology, evolution of scientific method has been of surpassing rapidity. Description was deserted when comparison became the order of the day. The mines of comparative morphology were in part abandoned when the cry was raised that experimentation was uncovering solid nuggets. The problems of biology are so numerous and so varied that no method of research can be permanently discarded. Observation can never fail to be the cornerstone of biology. The task of classification is only partly completed, even by those methods which were in use at the time when it was the major interest of naturalists. Taxonomy must profit by and ultimately incorporate all the pertinent facts unearthed by the newer methods of research. Comparison can never fail to yield results of importance. But all these methods may now be made more refined and exact by the introduction of mathematics applied to the description and analysis of quantitative data.

Two Points of Emphasis. In closing, I would like to return to the broader subject of mathematics in biology and to emphasize two points.

My first point is in the nature of a prophecy.

In the future, mathematics will have an increasing influence in determining the direction of research.

This is due not solely to the fact that the biometric formulas facilitate the solution of many problems, but also to the fact that after a certain stage in science is reached, calculation is to some degree capable of anticipating the results of experimentation. The value of the mathematician's prediction is well known to the physicist, the chemist, and the astronomer. As yet little progress in this direction has been made in biology, but I am glad to go on record as predicting that before many years have passed experimentation will be to a considerable extent guided by preliminary calculation.

My second point has to deal with a very different matter.

Elegance or form has always made a powerful appeal to the mathematician. As the biologist is forced by the inevitable progress of his science to occupy himself more and more with mathematical literature, its logic, terseness, and elegance of expression must have an influence upon his own standards of presentation.

Summary

Summarizing in a few sentences we may note that mathematics is driving into biology on two wide fronts.

On the one, physics and chemistry are by virtue of their influence upon biological research forcing biologists to take over the mathematics which is an indispensable part of these sciences. On the other, biometry is grappling with problems which are not readily amenable to experimental treatment.

The possible contributions of mathematics to biological science are too varied to be succinctly summarized. We must, however, record our entire disagreement with the dictum that mathematics is only a mill from which no more comes out than was originally put in. What are put in are raw data, the significance of which is obscured by all the perplexing irregularities due to morphological and physiological variation, to errors of random sampling, and to errors of measurement. What comes out is a series of mathematical constants and equations, epitomizing in mentally intelligible form the whole discordant mass of irregularities and smoothing them in a manner to bring out the underlying laws. To assert that the value of a biometric research is determined by the raw biological data is not altogether unlike measuring the value of a Titian by the grams of paint required to cover the canvas.

It has been the rare good fortune of Quetelet, Galton, and Pearson to initiate one of the great lines of advance in biology. These men will one day receive from biologists recognition as free and generous as their great service merits. As for the rest of that little handful of workers who have made up the biometric school, it has been the satisfaction of a few never to have stepped back from the guns during the long and discouraging years of biological indifference and opposition.

The ultimate recognition of mathematical biology is merely a part of that inevitable and irreversible evolutionary process by which biology is to take its place in the ranks of the exact sciences.

THE HUMANISTIC BEARINGS OF MATHEMATICS

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Mathematics and Humanism. In order to reduce the hazard of being misunderstood I will begin by giving some indication of the senses in which the terms Mathematics and Humanism are to be understood in the following pages.

Mathematics Defined.[1] After many centuries of endeavor it has become possible in recent years to define Mathematics with a high degree of precision and clarity. Mathematics may be viewed as a body of achievements or as an intellectual enterprise. I prefer to view it as an enterprise, and I define the great term in the following words: *Mathematics is the enterprise which has for its aim to establish Hypothetical propositions.* By a hypothetical proposition I mean one that either is stated, or admits of being stated, in the form, *p implies q*, where *p* denotes one or more propositions (called axioms, postulates, assumptions, or primitive propositions), where *q* denotes a proposition (commonly called a theorem), and where the verb *implies* is intended to assert that *q* is logically deducible from *p*.

It is common and often convenient to state a hypothetical proposition in the form: *If p, then q*. But here one must be on one's guard, for it is obvious that many propositions, though stated in this form, are not hypothetical. For example, the proposition—*if it lightens, then it will thunder*—does not mean to assert that the proposition, it will thunder, can be *logically* deduced from the proposition, it lightens. The test as to whether a proposition of the form—if *p*, then *q*—is or is not hypothetical is whether the assertor is intending, or not intending, to assert that *q* is logically deducible from *p*.

It is to be carefully noted that a hypothetical proposition is

[1] For a full exposition of this and kindred conceptions the reader may be referred to my book, *The Pastures of Wonder*, Columbia University Press.

true or false according as the asserted deducibility is or is not possible.

A mathematical proposition is a hypothetical proposition that has been established. And an "established" proposition is one that is so spoken of, so regarded, so treated, by all or nearly all experts in the field or subject to which the proposition belongs.

The foregoing conception of mathematics gains in clarity by viewing it side by side with the following conception of Science. This term ought, I am convinced, to be defined as follows: *Science is the enterprise having for its aim to establish Categorical propositions.* A categorical proposition is one stating that *such-and-such is the case*, regarding no matter what part or aspect of the actual world. A categorical proposition, no matter what its form, never asserts logical deducibility, or implication, but, as I have said, a hypothetical proposition always does.

By mathematical method I mean all available means estab-lishing hypothetical propositions. Of these means, one is s ereign, always absolutely indispensable. I mean *Deduction;* all other means are but auxiliary thereto, mere servants or helpers, never adequate in themselves. The law is: *No deduction, no mathematics.*

By scientific method I mean all available means for establishing categorical propositions. Of these means, one is sovereign, always absolutely indispensable. I mean *Observation,* all other means are but auxiliary thereto, mere servants or helpers, never adequate in themselves. The law is: *No observation, no science.*

In mathematics Deduction is supreme, observation and all other means subordinate.

In science Observation is supreme, deduction and all other means subordinate.

Types of Humanism. As for Humanism, it is especially important to indicate the sense in which the term is to be employed in this essay, for that fine old word is to-day used in such a variety of incompatible senses that, unlike the term mathematics, it cannot now be said to have a standard signification. And so there are humanisms and humanisms. It will, I think, be a helpful preliminary to signalize some of them.

There is, for example, the recently much-discussed Humanism which Professor Irving Babbitt has been endeavoring for two or three decades to formulate and foster by means of lectures, essays, and books, and which he and fourteen other representatives of his cult set forth a few months ago in the form of a symposium entitled *Humanism and America*, edited by Mr. Norman Foerster. In his editorial preface Mr. Foerster tells us that "Professor Babbitt has done more than any one else to formulate the concept of humanism," and that he is, in Mr. Foerster's opinion, "at the center of the humanist movement." It is, then, not strange that Professor Babbitt's rôle in the symposium is that of official definer. Mr. Babbitt tells us at the beginning of his "Essay at Definition" that definition is "indispensable." Naturally the reader is gladdened by the prospect of finding here an authoritative formulation of the proper meaning of the great term Humanism.

What, then, is Professor Babbitt's definition of that term? He states it in these words: "Humanists are those who, in any age, aim at proportionateness through cultivation of the law of measure." In point of form the definition is good, but what of its substance? No mere hitching together of such hazy verbal abstractions can convey any definite idea. Obviously the definition is sadly in need of interpretation. The entire essay may be viewed, and was doubtless intended to be viewed, as an attempt at such an interpretation. And what is the interpretation? It consists mainly of fragmentary description—of scattered bits of description—of what Professor Babbitt means by Humanism. Mathematicians need not be told that there is a radical difference between description and definition. Of Mr. Babbitt's scattered bits of description some are positive but most of them are negative. The most revealing of the positive bits are these: The humanist "may work in harmony with traditional religion," yet he says with Pope,

> Presume not God to scan;
> The proper study of mankind is man;

his central maxim is "Nothing too much"; like Milton he regards decorum as the "grand masterpiece to observe"; his final appeal is to intuition; the basis of the pattern he imitates is not divine but is "the something in man's nature that sets him apart simply as man from the other animals"; humanism manifests itself primarily, "not in the enlargement of comprehension and sympathy," but in "selection," in the imposition of "a scale of values"; like Matthew Arnold, the humanist "hates all overpreponderance of single elements"; he aims at approximating ever nearer and nearer to "perfect poise"; for the humanist the universe is sundered by a variety of old dualisms, the psychical and the physical, the subjective and the objective, the properly human and the subhuman or natural, the properly human and the superhuman or supernatural, "law for man" and "law for thing," the free and the determined; essential for him is the "higher will," here called the "higher immediacy," whose function it is to control the "lower immediacy"—"the merely temperamental man with his impressions and emotions and expansive desires."

Such are the positive bits of description, here assembled (not without labor) from various sections of Mr. Babbitt's essay. Most of them have long been familiar suggestions of the ideal and many of them are admirable. No doubt they help us somewhat to understand what it is that Professor Babbitt's definition of Humanism is intended to define. We must not be too sure, however, what these positive bits are designed to describe until Mr. Babbitt has interpreted them for us. In interpreting them he has supplemented them with many descriptive bits of the negative kind designed to tell us what Humanism is not. In the light of these negations we perceive that Mr. Babbitt's brand of Humanism is almost incredibly strict and exclusive.

Amazing, vast, and very impressive is the array of human interests, points of view, cults, activities, enterprises, personalities, enthusiasms, aspirations, dreams, that Mr. Babbitt, either explicitly or by implication and with the air of pontifical authority, excludes outright from the category of things humanistic. All monists (who deny or question the tenability of the old familiar dualisms and attempt to view the universe as a genuine cosmos somehow involving the unity of Nature and Man), all naturists (who believe that "out of the *earth* the poem grows like the lily or the rose"), all humanitarians (actuated and sustained by faith in the endless perfectibility of mankind), all romanticists, all determinists, all realists, all the philosophers who regard the One as a "concept" instead of a "living intuition," all the colleges and universities and other educational institutions that "proclaim the gospel of service," all pragmatists, all psychologists, all devotees of science, all specialists (except specialists in Mr. Babbitt's variety of Humanism); *all* of these and yet other kinds of the unworthy are rigorously excluded by Mr. Babbitt from his humanistic tabernacle. To admit men and women having any essential similitude to such as Epicurus or Lucretius or Nietzsche or Shelley or Shakespeare or Rabelais or the Walter Paters or the Rousseaus or the Walt Whitmans or the William Jameses or the Benedetto Croces or the Thomas Hardys would be to profane the sacred temple, and so they are debarred. Debarred also, by the clearest indicia of Professor Babbitt's conception of Humanism, are such creators of psychic light as Spinoza, John Locke, David Hume, Euclid, Newton, Einstein, Willard Gibbs, Lobachevski, Riemann, Gauss, Laplace, Lagrange, Charles Darwin, Herbert Spencer, Henri Poincaré, Louis Pasteur, to name only a few of the greatest of the great Unfit.

It is evident that Humanism, as conceived by Professor Babbitt, is too lacking in catholicity, in spiritual amplitude, in magnanimity, to attract any one except Mr. Babbitt, his disciples, and fellow symposiasts, who regard it as the sole remedy for healing the cultural maladies of the world, especially in America, and as the sole means for qualifying human individuals to represent worthily, in their life and work, the great potential dignity of Man. Some have called it "academic" or "strict" or "doctrinal" Humanism. It might be described, not inaptly, as supercilious, sectarian, pharisaical, arrogant. "It is," as Doctor H. S. Canby has said, "a very porcupine hunched up against our familiar world." Speaking of its central standard of literary excellence, Mr. Henry Hazlitt has written: "We are above all to judge a writer, not by his originality or force, not by his talent or genius, but by his decorum! That is, we are to praise him for a virtue within the reach of any learned blockhead."

Very different from the foregoing is the type of Humanism delineated and advocated by Charles Francis Potter in his beautifully written, sympathy-winning book *Humanism: A New Religion*—very different in content, in manner, and in spirit. For Mr. Potter and his kind, "Humanism is faith in the supreme value and self-perfectibility of human personality." It might be called human Humanism or nontheistic Humanism because, though it believes in man, it does not believe in a supernatural God. Mr. Potter's conception of Humanism as a religion is probably due to the fact that he was bred in, and for many years practiced, theology. His religion, however, has recently undergone a great change, for he now agrees with Ames that "religion is the consciousness of the highest social values" and with Haydon that "religion is the shared quest of the good life." Mr. Potter's religious Humanism is scientific in the sense that it looks mainly to science for help in more and more realizing "the twin visions"—the vision "of an ideal developed human personality" and the vision "of an ideal commonwealth made up of such personalities." But Humanism can no more be defined exclusively in terms of Religion than in terms of Art or Politics or Education or Literary Criticism or any other one among the great interests of man as man.

A third variety of Humanism is that portrayed in *The New Humanism* by Leon Samson. The book is bold, richly suggestive, frequently keen, notably omniscient, strangely visionary, and often flamboyant. Mr. Samson's so-called Humanism might be fairly described as loquacious Humanism, for, says he, "There is nothing that so unmistakably marks a man human as his capacity to talk"; or it might be called proletarian Humanism, for "the prized jewels of contemporary society will be turned to ashes when the proletariat lights the fire of life and love on the funeral candles of civilized culture"; or it might well be designated utopian or elysian Humanism, for it envisages a planet-wide community of humans who, having outgrown both war and work and the making and reading of books and all religions and morals and governments and all other historic or existing institutions, will thereafter devote their unbroken leisure ecstatically to endless conversation—to honest, original, infinitely varied and, of course, unfatiguing musical discourse by word of mouth.

The Proper Meaning of Humanism. It is hardly necessary to say that in dealing with the humanistic bearings of mathematics I shall have in mind a conception of Humanism vastly different from any of the foregoing varieties. It cannot be defined in terms of Mr. Babbitt's "decorum," "proportionateness," and "law of measure," nor in terms of Mr. Potter's excellent "religion," still less in terms of Mr. Samson's wholly fatuous proletarian dream. Indeed I shall not attempt to define it at all. For, as in the case of many another great idea—that of justice, for example, or wisdom or poetry or knowledge or truth or religion or art or love—its significance is too immense, embracing too much of life, to admit of being confined in a precise formula. But, though it cannot be neatly defined, it can be described well enough for the purposes of identification and recognition. In respect of brevity, clearness and comprehensiveness, combined, the best description I have encountered is in the following words of Mr. Walter Lippmann: Humanism "signifies the intention of men to concern themselves with the discovery of a good life on this planet by the use of human faculties." The Humanism indicated by that description is, in spirit, in aim, and in implicit scope, identical with the Humanism which, beginning in the fourteenth century, sprang into full life and greatly flourished throughout the fifteenth century as an essential part of the Renaissance, first in Italy and later in other countries of Europe. The term humanist, which then came into use, was applied to those men who, by their activity, proclaimed the full recovery of a very precious and very powerful human sense, one that had been lost and almost extinguished in the preceding long centuries of submission to external authority—I mean the sense that humans are, as such, endowed with the dignity of autonomous beings, potentially qualified by native inheritance to judge individually and independently in all the great matters of human concern and, by the exercise of their own faculties, to fashion their lives worthily.

That sense of personal autonomy is essential to the proper dignity of man and it is, as I have intimated, in the central core of Humanism. The fact is continuously manifest, and frequently becomes articulate, in the activity of the great humanists of the Renaissance. Let me cite one or two examples. One of the most illustrious humanists of the fifteenth century was Pico della Mirandola. In his famous *Oration on the Dignity of Man* he represents God as addressing Man in the following remarkable words: "The nature allotted to all other creatures restrains them within the laws I have appointed for them. Thou, restrained by no narrow bounds, shalt determine thy nature thyself according to thine own free will, in whose power I have placed thee. I have set thee midmost the world in order that thou mightest the more conveniently survey whatsoever is in the world. . . . Thou shalt have power to decline unto the lower or brute creatures. Thou shalt have power to rise unto the higher, or divine, according to the sentence of thy intellect." Note Pico's vigorous assertion of the autonomous nature of Man, and observe, too, how perfectly his utterance rimes with the following words of another eminent humanist of the time, Leon Battista Alberti: "Men can do all things if they will." Having in their hearts so living a sense of personal sovereignty it is no wonder that the great humanists of that period cast off the shackles of ecclesiastical authority, and it is no wonder that they were so eager in seeking, mastering, and emulating all that remained of the literature, philosophy, science, and art that had been created by the great humanists of antiquity.

Humanistic Education. I hope that I have now sufficiently intimated what it is that the term Humanism is to stand for in the following discussion. As most of those who will read this essay are professional teachers I will try to view my subject from the standpoint of an educator and will deal with it as an educational theme.

By humanistic education I mean education having for its aim to qualify human individuals to represent worthily, in their life and work, the great potential dignity of Man. In other words, I mean education characterized by the aim of qualifying human individuals "to discover" or—what is tantamount—to create "a good life on this planet by the use of human faculties." I say "on this planet" because in all times the great humanists have been sane enough to concern themselves primarily, if not exclusively, with mundane affairs, with means to excellence of life here upon the earth. The discerning reader will readily see that the two statements of aim are virtually equivalent.

It is obvious that in humanistic education we are concerned with the highest and most composite of genuine ideals. "Composite" because it embraces many other ideals which, though they are also genuine, are subordinate and auxiliary, for genuine ideals constitute a hierarchy of dignities. In this connection I cannot refrain from saying, what I have repeatedly said elsewhere and shall never miss an opportunity to say, that genuine ideals are not goals to be reached but are perfections to be endlessly pursued. Genuine ideals are like those mathematical limits whose variables approach them ever more and more nearly but never attain them. I know not how to condemn with sufficient severity that all too familiar philosophy, for it is now in much vogue, which counsels us to eschew genuine ideals on the alleged ground that, because they are unattainable, they tend to dishearten and devitalize. To hearken to that counsel is to turn away from the most powerful lures to excellence. For it is pursuit of unattainable ideals that has led to the great triumphs of the human spirit in every department of life. It is indeed the proper vocation of man.

In the theory of humanistic education it is necessary to distinguish between a human being and the mere follower of a human pursuit. As animate creatures inhabiting a world where we humans are obliged, like the animals, to win our lives from day to day, we all of us are, or are destined to be, in some sense, hewers of wood and drawers of water. And so we all of us require vocational or professional training. We cannot escape the necessity of being specialists of one kind or another. The ideal of such training, the ideal of specialism, is efficiency. But efficiency is not the ideal of humanistic education. For humanistic education aims at the development and the disciplining of the whole man. And the man building a bridge is immeasurably greater than the engineer; the man teaching the calculus is infinitely greater than the mathematician; the man cultivating fields is vastly greater than the farmer; the man painting a picture is incomparably greater than the artist. Humanistic education does not exclude the ideal of efficiency. What it disowns is the ideal of mere efficiency. The ideal of humanistic education is intelligence, emancipation, magnanimity: intelligence regarding the human and the non-human worlds; emancipation from every manner of trivial or sordid things, emancipation from provincialism, from fanaticism, from bigotry, from prejudice, from the multiform tyranny of fear; and magnanimity, largeness of mind and spirit, imagination enough and sympathy enough and reason enough and emotion enough and will enough to gain and maintain the lordly poise of a freeman amidst all the trials and frustrations encountered in a vast, complicate, perplexing world.

Great Permanent Facts of Life and the World. It is obvious that humanistic education aims to orient and discipline our human faculties, not with special reference to the technical requirements of any given pursuit, no matter what, but with reference to all the great permanent massive facts of life and the world. A little reflection suffices to show that there are such facts.

One of them is the fact that every human being has behind him an infinite past out of which he has come and which contains for his guidance and edification the records or the ruins of all the experiments that man has made in the art of living in the world. It follows that humanistic education will not neglect the disciplines of the history and the literature of antiquity.

Another of the great abiding massive facts of life and the world is the fact that we humans are constrained by forces beyond our control to live, not in isolation, but in human society; that we are literally born members of a thousand teams with which we must learn to coöperate in some measure or perish. It is, therefore, evident that humanistic education is bound to provide discipline in political science, in social science, in ethics, and in jurisprudence.

A most impressive member of the group of great permanent massive facts of life and the world is the ubiquitous presence of Beauty. Beauty is the most precious and the most vitalizing element in the universe. More than aught else it is beauty that not only makes life worth living but makes it possible; for if by some spiritual cataclysm all the beauty of nature and all the beauty of art and all the beauty of thought were suddenly blotted out, our human race would quickly perish by depression of spirit owing to the omnipresence of ugliness. Consequently humanistic education will fashion itself in large measure by the consideration that those who are to be qualified for the discovery or creation of a good life must needs have taste in the arts of men and an awakened sensibility to the marvelous natural beauties of land and sea and sky.

Mathematics and the World of Ideas. Among the momentous facts with reference to which it is the function of humanistic education to orient and discipline our faculties I have now, finally, to signalize the stupendous fact denoted in German by the term *Gedankenwelt*—the world of Ideas. For equipping one to deal with ideas *as such*—to deal with them, that is, in accord with the laws of thought, in accord with the standards of rigorously sound thinking—there is but one available discipline, and it is that of logic and mathematics. In deference to conservative usage I have said logic *and* mathematics, though the maturest critics regard the two as one. It is evident that to be set in right relation to the world of ideas, though it is not alone sufficient, is certainly necessary, to qualify one to represent worthily the proper dignity of man. We are thus bound to say that, for humanistic education, logic and mathematics constitute not merely a useful discipline but one that is indispensable.

Attitudes in Mathematical Study. The fruits of that discipline naturally vary with the attitude of the student in pursuing it, and the student's attitude may be any one of three. He may, that is, pursue mathematics for its own sake or for the sake of its uses and applications or for the sake of what I shall call its bearings.

One who pursues mathematics for the sake of mathematics is sustained by its charm. Having gained some knowledge of it, he craves yet more, and what he has gained at any stage equips him for further gain. If he be a research mathematician, the investigations he makes lead him to further investigations, and so on endlessly. The attitude of such a student reminds one of the farmer who, when asked why he raised so much corn, replied, "In order to feed hogs," and when asked why he wished to feed so many hogs, replied, "In order to buy more land," and when asked why he desired more land, replied, "In order to raise more corn." I am not condemning the attitude, far from it, but merely indicating it. This attitude of the mathematician is indeed well justified by two considerations. One of them is that he has great joy in the game, and any one who has felt the joy knows how sustaining it is. The second consideration is that, if mathematicians did not pursue the subject without regard to its applications or uses, then, when mathematical doctrines are needed as instruments in the development of other scientific subjects, the required doctrines would not be in existence. Moreover, one of the important lessons of history is that doctrines created for the mere joy of creating them, created, that is, without regard to any question of applicability, are sooner or later found to be applicable and thus acquire a secondary type of justification. A familiar example of this very significant fact is afforded by the theory of conic sections, which was essentially worked out for the pure intellectual joy of it long before it found application in astronomy and navigation. A more recent and even more striking example is that of the frightfully complicate Theory of Tensors, established by Riemann and Christoffel long before the "idle theory" became, at the hands of Einstein and his fellows, the "backbone" of the General Theory of Relativity. In this connection I must repeat one of the delightful stories told of J. J. Sylvester when he was professor of mathematics in the Johns Hopkins University. One day as he was crossing the campus he encountered a notably practical-minded colleague, who said to him: "Professor Sylvester, what subject are you lecturing on this term?" The great mathematician replied: "On the Theory of Substitution Groups." "What," asked the practical man, "is the *use* of that theory?" "I thank God," said Sylvester, "that so far as I know it hasn't any." Sylvester was at heart a poet, galvanized to the highest mood by the beauty of pure thought. For him mathematics was "the Music of Reason."

Very different, ordinarily, is the attitude of a student pursuing mathematics in preparation for the practice of engineering. And the same may be said of one pursuing mathematics in preparation for a career in physics or astronomy or chemistry or statistics or economics or some other scientific branch in which mathematics has been found serviceable as an instrument of research. For such students mathematics is justified, not by its inner charm, but by its applicability. They regard the subject not so much as a branch of knowledge as an indispensable tool. The attitude of such students requires no defense. It sometimes happens, however, that such a student is inclined to depreciate one of the Sylvester type. The fact is well exemplified by Fourier who, as is well known, was devoted to the applications of mathematics to physical problems and especially to the theory of he[illegible]o much so that he reproached Jacobi for not devoting his magnificent abilities to similar problems. The reproach called forth from Jacobi a justly famous and very significant reply: "A philosopher like Fourier," said Jacobi, "ought to know that the unique end of science is the honor of the human spirit, and that a question respecting number is quite as pertinent thereto as a question respecting the physical world."

Humanistic Value of the Foregoing Attitudes. I shall not tarry here to argue at length the humanistic value of mathematics when pursued for its own sake or when pursued for the sake of its applications. With regard to applications, when I think of the immeasurable service rendered by mathematics in the countless ordinary affairs of the workaday world, in the developments of many branches of science, and in the invention of marvelous means for the conquest of space and time, I can hardly imagine any one dull enough to deny that in these ways mathematics has greatly contributed to the "creation of a good life on this planet by the use of human faculties." As for those great mathematical creations that owe their existence, not to any uses they may serve, but to the pure joy which their creation and contemplation yield, it is safe to say that hardly any other human achievements better demonstrate the dignity of man. To glance at a single phase of the matter, what can be found in the whole history of thought more humanistically edifying than the story of the evolution of the number-concept, from the rudest beginnings in primitive mind, long before men had learned even the first steps in the process of counting, to the great number-creations of the modern world? The concepts of Integers and Fractions, of Cardinals and Ordinals, of Positives and Negatives, of Rationals and Irrationals, of Reals and Imaginaries, of Algebraics and Transcendentals, of Finites and Infinites, these great concepts viewed with the occasions of their rise, with their struggles for existence, their ultimate triumphs over stubborn opposition, their persistent hardy growth through the centuries, their countless diversifications and subtle refinements, the infinite network of their interrelations and their manifold, always increasing, practical and theoretical uses and applications, afford a series of scenes that, for any one who has once contemplated them, constitute a truly unforgettable and inspiring panorama of the march of mind.

Mathematics One of the Humanities. It remains to consider the attitude of one who pursues the study of mathematics, not for the sake of mathematics, nor for the sake of its uses and applications, but for the sake of its *bearings*. By the bearings of mathematics I mean the relations of mathematical ideas, processes, and doctrines to such great human concerns as are not, strictly speaking, mathematical. One who pursues the study with a view to its bearings can hardly fail to discover that, by virtue of its humanistic significance and worth, mathematics is entitled to high rank among the great Humanities. For what are the subjects that are best entitled to be listed among the humanities? It can hardly be doubted that the answer ought to be this: Those subjects have the best claim to be called humanities which best serve to reveal the nature of our common humanity and best serve for the guidance of our human life.

It can be readily shown, I think, that, according to the double criterion just stated, the claim of mathematics to be regarded as one of the humanities is unsurpassed. Let us examine the matter a little. We must begin by asking: What is the chief mark of man as man? What is the defining quality or character of the essential nature of our common humanity? What is it that serves best to discriminate human beings from all other kinds of living creatures? The answer, I think, is this: The chief characteristic mark of man as man is what Count Alfred Korzybski in his book, *The Manhood of Humanity*, has called the time-binding capacity of human beings. The fine term denotes that highly composite faculty in virtue of which each generation of mankind is enabled to employ the accumulated achievements of the preceding generations as capital for the production of yet greater achievements, so that, as the generations succeed each other, science begets better science, philosophy better philosophy, art better art, jurisprudence better jurisprudence, ethics better ethics, religion better religion, invention better invention, and so on. By that composite faculty man is set apart from the animals and the plants. It is the secret of the progressability of our human kind. It is the civilizing energy of the world.

Where does the time-binding power of man make itself manifest? Obviously it manifests itself in the development of all great subjects and human enterprises. I now ask: Where is this defining mark of man revealed most clearly? It is most clearly revealed in mathematics, for in the continuity of the progressive development of mathematics, running from remote antiquity down through the centuries and flourishing to-day as never before, the time-binding power of the human intellect is not only revealed but revealed in its nakedness. I contend that, by this superior disclosure of the characteristic nature of our common humanity, mathematics conclusively vindicates its claim to distinguished membership in the assembly of the humanities.

If we turn now for a moment to contemplate mathematics regarded as a guardian and guide of our human life, we shall find that the foregoing conclusion is abundantly confirmed. As every one knows, human activity presents certain great distinctive types. One of the greatest of these types is that which we call logical thinking, not merely thinking but logical thinking, the generating of precise ideas, the combination of them, the relating of ideas in the forms of judgments and propositions, the uniting of propositions to form doctrines for the enlightenment of the human understanding and the guidance of human conduct. Every one knows or ought to know, for the fact is sufficiently obvious, that above each of the great types of human activity there hovers a shining ideal of excellence—a muse, if you will, or guardian angel wooing and beckoning us upward along the steep endless path toward perfection. Now, what is the muse or the angel that lures and sustains our efforts in logical thinking? The name of the muse is familiar—it is Logical Rigor, the name of an austere goddess demanding, though never quite securing, absolute precision; demanding, though never quite securing, absolute clarity; demanding, though never quite securing, absolute cogency. Is this unattainable ideal of absolute rigor a valuable one in the conduct of human life? The answer, which cannot be too emphatic, is this: In every department of life where thought is required, the standard of logical rigor is so valuable that just in so far as our thinking departs from it, our discourse sinks down toward the level of mere chattering—the noise and gabble of our prehuman and subhuman ancestors.

Why is it that the standard of logical rectitude is so clearly revealed in mathematics? And why is it that mathematics succeeds so famously in approximating conformation to the standard? The secret lies in the method of mathematics—the method of carefully selected and clearly enunciated postulates, of sharply and completely defined concepts, and of painstaking deductions or demonstrations. Because there is no field in which a worker can escape the necessity of making conscious or unconscious use of postulates, nor the necessity of formulating definitions and of attempting deductions and demonstrations, it is evident that mathematical procedure furnishes a model for the guidance of criticism of all discourse of reason, no matter what the subject or field to which the discourse pertains.

Mathematics and the Universal Concerns of Man. The considerations thus far advanced, though fundamental and decisive regarding the title of mathematics to be listed among the humanities, are far from being all that might be adduced. One who open-mindedly contemplates the humanistic bearings of the subject will be led sooner or later to see that, as I have said elsewhere,[2] "*Every major concern among the intellectual concerns of man is a concern of mathematics.*" No doubt that statement will seem to some to be extravagant. Yet the statement is true and the truth of it ought to be made known to all. Let me briefly submit a few justifying considerations.

Every one knows that among the most impressive facts of our world is the great fact of Change. The universe of events, whether great or small, whether mental or physical, is an endlessly flowing stream. Transformation, slow or swift, visible or invisible, is perpetual on every hand. But events are interdependent, so that change in one thing or place or time produces changes in other

[2] See *Mole Philosophy and Other Essays*, E. P. Dutton and Company.

things and places and times. With the processes of change every human being—moron, mediocre, or genius—must deal constantly or perish. The processes of change are not haphazard or chaotic, they are lawful. To deal with them successfully, which is a major concern of man, it is necessary to know their laws. To discover the laws of change is the aim of science. In this enterprise of science the ideal prototype is mathematics, for mathematics consists mainly in the study of functions, and the study of functions is the study of the ways in which changes in one or more things produce changes in others.

We are here in the presence of another bearing of mathematics upon a major concern of man. I mean our human concern to ascertain what things, if any, are permanent in the midst of change. Human beings desire to know what things, if any, abide. We wish to know what things, if any, may be counted upon. In this great quest of permanence in the midst of mutation is found the unity of science, philosophy, art, and religion, for it is the sovereign concern of them all. Is it a concern of mathematics? To find the answer one has only to glance at the immense mathematical literature embodying the truly colossal doctrine of Invariance.

Next consider the great subject of Relations. Such terms as spouse, husband, wife, father, mother, parent, child, king, subject, president, citizen, partner, enemy, friend, greater, less, better, worse, and so on and on, are familiar examples of relations. Whoever examines the matter will be astonished to find that most of the words in any language, either directly or indirectly, either explicitly or implicitly, denote relations. Each thing in the world has named or unnamed relations to everything else. Relations are infinite in number and in kind. Being itself, said Lotze, consists in relations. Science, said Henri Poincaré, cannot know "things" but only "relations." To be is to be related. To understand is to understand relations. To have knowledge is to have knowledge of relations. It is evident that the understanding of relations, the gaining of relation-knowledge, is a major concern of all men and women, whether they are aware of it or not. Are relations a concern of mathematics? They are so much its concern that able critics have thought it possible to regard mathematics as having relations as its sole concern.

I have thus far said nothing explicitly regarding morals and religion. Undoubtedly these are among the major concerns of man. A word regarding ethics. We know that, in any fairly stable community, no matter how primitive, no matter how civilized, there gradually rise and ultimately prevail certain sentiments regarding "right" and "wrong," regarding "good" and "bad," regarding what "ought" to be, and "ought" not to be, in human conduct. These sentiments get themselves expressed in the forms of maxims or propositions. The propositions are regarded by all, or by most, members of the community in question as embodiments of ethical knowledge or truth. The body of propositions is an ethical system grown out of experience. Such systems vary from community to community, and from period to period in the life of a given community. What service can mathematics render in connection with such a system of ethics? The question has been well answered by Jacques Rueff in his excellent little book, *Des Sciences Physiques aux Sciences Morales*. He has here shown that, in the case of any such system, it is always possible to find a set of principles or postulates from which the propositions of the system can be deduced as consequences in the mathematical manner. In this way an ethical system rises from the level of experience to that of a logically organized doctrine. The transformation is one imperiously demanded by the human intellect. Moreover, by such a transformation an ethical system is shaped for criticism. And this is well, for, as Cousin long ago said, "La critique est la vie de la science." I should add that Rueff's book has been translated into English and published by The Johns Hopkins University Press under the title, *From the Physical to the Social Sciences*.

The bearings of mathematics upon religion are treated by another essay in this volume. I will, therefore, content myself with a single relevant observation. It is that the concept of infinity which is involved in the great question of immortality, is dealt with in mathematics, but not elsewhere, in strict accord with the standard of logical rectitude.

In the light of what has been said and suggested in the foregoing discussion it is abundantly evident, I believe, that, among the agencies for qualifying human individuals "to create a good life on this planet by the use of human faculties" or to represent worthily, in their life and work, the great potential dignity of Man, Mathematics is unsurpassed.

MATHEMATICS AND RELIGION

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The Bonds Between Them. It is one of the tendencies of the mind to look upon its own major interest as the focus of all knowledge. The mind tends to see analogies that are, at best, remote; to magnify the influence which its own favored science exerts upon all other branches of knowledge; and to feel that it detects bonds which do not exist. The poet, for example, sees in the Book of Genesis a magnificent prose poem, the uplifting power of which vanishes when he thinks of it as a treatise on natural science. The mystic sees in the Old Testament a field of what he feels is mysticism, and he reads into it a harmless obscurity that pleases him and has the merit of injuring no one else. The Christian apologist finds in its inaccuracies, as that π (*pi*) equals 3, the errors of some ancient copyist, instead of frankly recognizing that the one who wrote that particular verse simply used the everyday common value adopted by the people of his time. The mathematician may, for a similar reason, tend to exaggerate remote analogies and to assert a closeness of relationship between his own field of interest and that of the theologian that is, in fact, very attenuated. In speaking of this relationship, therefore, one must always be on his guard against trying to see the invisible or to imagine that which has no existence. So much for the initial objection of those who look upon knowledge as made up of separate and distinct domains.

The bonds uniting mathematics and religion have often been considered, and many have been the monographs written and the words spoken upon the subject. The trouble is that the attempts have generally concerned the theologian on the one side and the mystic with some knowledge of elementary mathematics on the other. They have only rarely been made by either the seeker after the good, the true, and the beautiful in religion, on the one side, or the constructive mathematician of genius on the other. Even when the conditions have seemed favorable, however, as when Newton wrote on religion, the result has been more lamentable than convincing or inspiring. Cauchy was a mathematical genius but a religious bigot, and many a religious leader in the Middle Ages looked upon mathematics as an invention of Satan.

Hence it is evident that no brief essay is likely to contain much that is new or to have an influence that reaches beyond a small circle of readers. Nevertheless there is some advantage in the intimacy which such circles afford, for a writer can feel that a relationship is created which allows him to consider that he is among friends to whom he is bound by ties of common interest.

Terms Considered. If by the term "religion" is meant sectarianism, or even Christianity, or Buddhism, or Brahmanism, or Mohammedanism alone, then this essay will mean but little. If by the term "mathematics" is meant nothing but number mysticism, or hyperspace, or a belief in the infallibility of any single dictum of the subject as we now know or do not know it, then these remarks will mean even less. But if the reader belongs to that increasing class of those who have a general knowledge of and a sympathy with both religion and mathematics in the large, then it may have at least the value of suggestiveness.

All religious writings tend to a mysticism which arises from ignorance or from the inability to explain the inexplicable. As part of this mysticism is the mystery of numbers. The three primes in the common number realm of primitive peoples—the 3, 5, and 7—enter into the rituals of practically all religions, and it would be an interesting but rather profitless task to write a history of any one of them. The material is sufficient for a labor of many years, but after it was carefully sifted, the result would be that these numbers are mystic simply because they are prime, and that for this reason alone they have been forced into all domains of religious mysticism.

What concerns us, this little circle of readers, however, is something quite different, namely, the influence of elementary mathematics upon the religious instincts of youth; an influence unconsciously or at least unobtrusively stimulated by any teacher without bigotry, by one having no wish to indulge in propaganda and not possessed with that fatal defect of hallucination which leads mankind to believe that it sees or hears or feels that which has no being.

The Infinite. First, mathematics soon leads us to a feeling that the Infinite exists. The inquisitive child shows this when he asks his teacher what is the largest number; and the teacher shows it in her inability to reply. This feeling grows more impressive when the child becomes the youth and studies any elementary series, even the summation of n terms of the geometric or any other type. It increases when he studies geometry and wonders what happens to the sum of the angles of a triangle when the vertex is "carried to infinity," and when he asks the teacher what infinity means. It increases when he studies simple trigonometry and finds that, as an angle approaches 90° the tangent approaches infinity, suddenly becoming minus infinity when it passes through the right angle. It increases when, if ever, he becomes a scholar in mathematics, and deals with the infinities of higher orders, with transfinite numbers, with the several plans of representing infinity graphically, and with the infinity of time and space, or with the finiteness of each. And finally, when he measures the known universe, or universe of universes, and thinks in light years (the distance that light travels in a year), and finds that the distance across explored space may be 400,000,000 of these light years, and lets his imagination carry him to the verge of this space and leads him to wonder about that which lies beyond—then the mystery becomes overpowering. He has pushed back the clouds of ignorance only to see that his own ignorance has become more and more hopeless, and that science leaves him helpless in the presence of a new infinity. The childish boast that we will believe only what we see, the most childish of all our feeble assertions of our faith in our puerile strength, avails us not. Mathematics has lured us on, and at the last we feel more helpless than ever, because we have come to see how full of awe we are in the presence of the awful Infinite.

The Changing Bases. Again, the youth need not even reach the legal age of the adult before another flood of mysteries tends to engulf him. He is taught in mathematics that certain postulates are sacred and that he must not question them. In religious instruction he is taught the same. In mathematics he will be encouraged by any honest and capable teacher to see that certain postulates are not always true: in religion he may be condemned if he queries certain others. In general, many teachers in each domain display a kind of fear of honest inquiry, a fear based either upon ignorance, or upon faith in tradition. No field of mathematics need fear searching inquiry, and no religion or sect need fear the scientific study of its essential nature, however much it may fear a study of the nonessentials which have accumulated through the ages. In mathematics it is evident that each of these series has the same number of terms, however far we go:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \ldots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \ldots \end{array}$$

for each term of the second is formed from the term of the first that lies just above it. If, therefore, the number of terms of each is unlimited, the number in each case may be said to be the same. But the second series is a part of the first, consisting of every other term. Hence, in this case, the part is equal to the whole. The illustration is a common one, and equally common is the one which shows that the infinity of points in one line is the same as that in a line twice as long, or half as long, or one-tenth as long, or a million times as long. Therefore the youth in school readily comes to the stage at which he sees that postulates that are valid for the small field in which he has lived, and that are necessary in such a domain, cease to be so when he faces the Infinite. A postulate is an assumption of validity in some special region; but we outgrow postulates as we outgrow clothes, whether in mathematics, in physics, in behavior, or in any other domain, substituting new ones which appear to have validity in the new region of thought in which we find ourselves. When Einstein made known his theory, it did not destroy Newton's postulates in gravitation, or his laws. These are valid up to a certain point, or at least are practically workable. He simply took the next step. When Lobachevsky and Bolyai proclaimed their theory of parallels, they did not destroy Euclid's postulates; they simply assumed another set and worked out a new lot of conclusions. In ordinary finite space, Euclid's geometry is a workable one; in space in general the other has advantages. Euclid tacitly assumed that space was everywhere alike and that a straight line, however far produced, never returned into itself; modern writers tend to assume that space is curved and that what we think of as a straight line is like a great circle on a sphere, always returning into itself. Mathematics simply says, "If this, then that"; it does not say, "This is eternally true, therefore that is eternally true." It never fears to have a search made into its fundamentals; science never fears it; and no religion, whatever it may be, need fear it. An ignorant teacher may fear that Euclid has been killed; but the great Alexandrian has been killed a thousand times by over-ambitious teachers, and yet he is more vigorous to-day than ever—vigorous because the bases of geometry have been more firmly fixed through a search into its fundamentals, and because the nature of proof is more clearly known. It is the nonessentials that go; the essentials stand. Modern religious thinking leads to the same conclusion—that it has nothing to fear from honest study; if its nonessentials go, the essentials will stand the more firmly.

The Play of Imagination. Like physics, astronomy, religion, art, and poetry (whether these are separate regions of thought need not concern us at present), mathematics offers a field for the play of the imagination that is tending more and more to be cultivated. Take such a simple figure as the triangle ABC. If we place a point X on the base AB, the figure may be considered as a four-sided one, $AXBC$. What has now happened to the sum of the angles of the figure? Any pupil in geometry can readily answer the question. What happens if X moves upward, so that the figure has a reëntrant angle? Then let X roam as it will, now resting on BC, now resting at B, now passing through BC, now going to "infinity"; now "passing through infinity" and returning from the other side—and so on. What a chance for the play of imagination is here! And has not this play its analogue in the speculations to which the searcher after the great things in religion is led as he comes to consider the relation of the finite to the Infinite? For what the youth in his study of this simple figure has found is another illustration of the permanence of laws, the permanence of truth, the confidence that may be his in his proofs of propositions relating to what we know as "Euclidean space." It is a wonderful thing to come in contact with something, however insignificant, that is "the same, yesterday, to-day, and forever," even in a hypothetical field. If the speculations of youth in such a field are noble, how much nobler are those in fields more extended and more vital!

Space and Time. Mathematics leads inevitably to a consideration of the nature of space and time. The consideration may seem trivial to us, but that it exists at all is significant. Where does space end? Has it any end? If it curves, through what?

Is it through a fourth dimension, which scientists now come to consider a commonplace idea? If so, what other spaces are there and what is their nature? Not without value are the rather childish considerations of Flatland and its inhabitants, and of the way in which a four-dimensional being may look at us and at the lives we live. If this raises the question of other spaces, of other universes of universes, the speculation, immature though it may be, has value. It places religion in a new light, it sets new bounds to human impotence, and it takes away the boastfulness of a mind characterized by "arrested development." The same influence comes with respect to time. Is it, like space, a closed affair, returning into itself? Is it, like length, a dimension—the fourth dimension? We can point to the north, but can we point to to-morrow? Time has the elements of a dimension but we are too three-dimensional-minded to point to the direction it takes.

Algebra, like geometry, leads us to similar speculations. The equation $2x + 3y = 6$ is represented geometrically by a straight line in a flat surface (a space of two dimensions); $2x + 3y + z = 6$ is represented geometrically by a plane in our space of three dimensions; but what about the equation $2x + 3y + z + 4w = 6$? Have our dimensions given out? Should it be a solid in a space of four dimensions? And if so, where shall we end in our speculations? Into what kind of a super-cosmos are we being led? People say, "I will not believe in God," but they believe firmly in Nature, and most of them have the faint remnants of a belief in signs, in omens, in luck, and in looking at the new moon over the right shoulder. They will not believe in any possibility of a world beyond, but they will see the entire possibility, and at present they believe in the probability, of a dimension beyond our own. It is a curious situation, this religious skepticism, and it would seem that it would tend to vanish if the theologian were not afraid of honest search after the fundamentals of religion. This, at any rate, has been the result in the fields of mathematics and of science.

Obscurity of Language. Much of mathematics and also of religion is obscured by the language used. Mathematicians in the sixteenth century spoke of negative numbers as "fictitious" but as soon as it was found that they could be represented graphically and used physically, they ceased to be such. They then became no more fictitious than a fraction, for we can neither look out of a window $\frac{1}{2}$ of a time nor look out of it -2 times. Each number may be called artificial, but neither is fictitious. Similarly, the square root of a negative number has long been called "imaginary," but as soon as it was represented graphically and used physically it ceased to be any more imaginary than a fraction or a negative number. Religion has met with the same difficulties. Much of the trouble that young people have in comprehending it is due to the presence of terms and concepts that are entirely irrelevant to the essence of the subject. Eliminate these, or at least make them more concrete, and this difficulty will be lessened or will even disappear, as it has in mathematics. If we would cease teaching a considerable part of current algebra and concentrate on the great features of even elementary mathematics, pupils would get much more out of it. If we should do the same for religion, the result would be equally beneficial.

Some Effects of Mathematical Study. A very good friend of mine, and one for whom I have great respect, is continually demanding, "Will anyone tell me why the girl should study algebra?" My answer would be that she should not study it at all if it is to be taught to her as it was taught to him, or as he and I may one time ourselves have taught it. After we have cut off all the useless traditional growth that encumbers it, she should study it for two or three very good reasons, one of them being that she has as much right as a boy has to feel her position in the universe, to be led to consider such concepts as time and space, to appreciate the nobler side of mathematics and to experience the feeling of the grandeur of space that it opens to youth. Then I would ask my friend in return, "Will anyone tell me why the girl should know anything about the nobler side of religion?" The two questions are complementary. Perhaps he might say that dish-washing is more noble than either mathematics or religion, and that to play bridge is more important than to think of what the infinite in algebra or in religion means to her or to her brother; but does he really think so?

Through elementary mathematics the youth finds for himself that certain laws are eternal, that in the presence of the Infinite some of his early beliefs must be abandoned, that what he once believed to be a fiction is as real as life itself, that there may be other spaces than ours, that our space may be curved just like the thin space on a sphere, and that death has no effect upon the eternal truth that the square of $a + b$ is $a^2 + 2ab + b^2$. He sees that quartz always crystallizes as a hexagonal prism joined to a pyramid and that the bee's cell patterns after it, and has done so for millions of years and will do so for billions more. With the unseen tendrils of his mind he grasps the Eternal. No book, no priest, no teacher, no authority has led him to see that the certainties of mathematics have helped to conquer what he considered his certainties of childhood. The essential features of religion offer no difficulties that differ greatly from those which he may easily conquer in the domain of the Mother of Sciences.

Do children get this from their mathematics? Not when led by teachers whose minds and interests have never seen the light—and so with their courses in education, in science, in history, in religion, and in art. But a new era in the teaching of mathematics is dawning, an era in which tradition gives way to a nobler conception of what all the sciences mean in relation to one another and in relation to the higher ideals of humanity—the fine arts, the cultivation of a taste for better literature, the social life and the comforts of the world, and the religious instincts of our race.

THE MATHEMATICS OF INVESTMENT

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PART I. INTRODUCTION

Extent of the Discussion. Along the backbone of the business world we find those problems which involve simple interest, compound interest, and the applications of compound interest in the theory of both annuities certain and contingent annuities. The word *annuity* here refers to any sequence of periodic payments. The theory of interest and annuities and their applications is referred to as the *mathematics of investment*, or the *mathematics of finance*; it is a part of the more extended field of actuarial science. The mathematics of investment, as distinct from mere arithmetic, is at the foundation of scientific banking, accounting, bond practice, all forms of activity involving the investment of money and the discharge of debts—particularly the discharge of debts by sequences of periodic payments, life insurance, and life annuities. In the present chapter, we shall discuss the theory and applications of *simple interest, compound interest*, and *annuities certain*. We shall not consider the theory of contingent annuities, and their applications, which occur mainly in a treatment of life annuities and of life insurance. This aspect of the mathematics of investment is not of such general appeal as is the rest of the subject, and, moreover, its treatment presents essential theoretical and notational difficulties.

Mathematical Prerequisites. A minimum satisfactory mathematical background for the study of the elements of the mathematics of investment consists of arithmetic, and one and one-half years of high school algebra. Familiarity with the computational aspects of logarithms is very desirable, but is by no means an essential part of this background; it is merely convenient to simplify the arithmetic involved by using logarithms, or, better, by using a computing machine. In any rational treatment of the mathematics of investment which does not introduce artificial mathematical difficulties merely as theoretical toys, there is no need of the use of logarithms in solving exponential equations, or other advanced topics in the theory of logarithms. In the present chapter, logarithms will be referred to only in brief discussions where an effort will be made to properly gauge their usefulness, or the necessity for their presence. Knowledge of the formula for the sum of a geometrical progression is the only item of the theory of progressions which will be needed, and this formula will be used only once in the chapter. This rather extended discussion of the rôle which logarithms and progressions will play has been given in order to dispel at once an all-too-prevalent notion that the mathematics of investment is heavily dependent on progressions and logarithms. A proper foundation for the mathematics of investment consists mainly of good arithmetical skill, that familiarity with the use of literal numbers and algebraic manipulation which results from one a· l one-half years of algebra, and such maturity of experience as is possessed by students of the senior high school, or higher levels.

Historical Background. The mathematics of investment developed very early, along with the use of interest in financial transactions. From the year 1202 onward, books on mathematics included prob' ms concerning both simple interest and compound interest. At an extremely early stage we find problems of great difficulty, as judged from the standpoint of the early mathematician who had neither the theory of logarithms nor the extensive modern interest tables at his disposal. Thus, Fibonacci (1202) proposes the following problem:[1]

> A certain man puts one denarius at interest at such a rate that in five years he has two denarii, and in every five years thereafter the money doubles. I ask how many denarii he would gain from this one denarius in 100 years.

In Tartaglia's *General Trattato* (1556), we meet the following problem,[2] whose solution by Tartaglia, with his inefficient mathematical tools, demanded great ingenuity:

> A merchant gave a university 2,814 ducats on the understanding that he was to pay 618 ducats a year for nine years, at the end of which the 2,814 ducats should be considered as paid. What interest was he getting on his money?

[1] See Vera Sanford, *A Short History of Mathematics*, p. 130. Houghton Mifflin Company, 1930.

[2] See Vera Sanford, *op. cit.*, p. 136.

In fact, we can say that essentially all of the small amount of theory on which the mathematics of investment is based has been known for centuries.

The Unfamiliarity of the Subject. The mathematics of investment consists of very little theory and of very much application, and extremely interesting and practical application at that. Beyond the level of elementary arithmetic and intuitional geometry, we should class the mathematics of investment as practically the most elementary field of applied mathematics. In it, we find that arithmetic and (to the astonishment of the elementary student) even algebra find continual use. In spite of the long history of the mathematics of investment, its elementary character, and its great practical importance, American schools have been slow to move any appreciable part of the subject down to the school level where it might well be brought in. Until comparatively recent years, one could not refer to the mathematics of investment even in an audience of professional mathematicians without fearing that most of one's listeners were practically uninformed in regard to the applications of the subject at hand. Up to fifteen years ago, very little effort was made in American colleges to teach a specialized course in the mathematics of investment except to that extremely small group, present in only a few universities, which was receiving the training necessary for the actuarial side of life insurance. The main difficulty in presenting the mathematics of investment to elementary American students was the lack of a proper textbook. This difficulty was successfully overcome for the first time in 1913, when Professor E. B. Skinner of the University of Wisconsin made an outstanding contribution to mathematical pedagogy by the publication of the first edition of his *Mathematics of Investment*.[3] Since then, a great variety of other texts of an elementary nature have appeared, offering various modifications of the course as it was originally organized by Professor Skinner. With the aid of the many excellent texts on the subject, the mathematics of investment is now taught successfully to college freshmen on an extensive scale, mainly to those who are specializing in business administration, but also to large numbers of students in the general college courses. On account of the comparatively recent appearance of the mathematics of investment in the college curriculum, and the almost entire absence of

[3] Published by Ginn and Company.

any treatment of it at the secondary level, the discussion of the present chapter will presuppose no knowledge of the subject outside of the familiar facts about simple interest and compound interest.

Outline of the Discussion. Part II of the chapter will be devoted to a discussion of certain aspects of simple interest, and of the related topic of simple discount. Admitting that the material referred to in Part II is present in high school arithmetic, Part II aims to orient this elementary section of the subject with respect to the more advanced portions. The author believes that such orientation could with advantage be emphasized in the teaching of arithmetic.

Part III will be devoted to compound interest, with emphasis on simplification of method. Such simplification comes from (1) the use of the interest period instead of the year as the fundamental unit of time in the formulas, and (2) systematic use of interest tables and interpolation in them, in place of logarithmic methods.

Part IV will be concerned with a derivation of the fundamental annuity formulas, and their application in various types of problems. The length of Part IV, relative to Parts II and III, does not properly indicate the outstanding importance of Part IV in the mathematics of investment. Lack of space for descriptive material makes it impossible to present more than a brief indication of some of the many applications of annuities certain.

Part II. Simple Interest and Simple Discount

An Algebraic Treatment of Simple Interest. At any school level beyond that on which the student has dealt with linear equations, the treatment of simple interest and its applications should be definitely tied to the familiar equations:

$$I = Prt; \qquad (1)$$
$$S = P + I; \qquad (2)$$
$$S = P(1 + rt). \qquad (3)$$

In these equations, P is the original principal, r is the interest rate, expressed as a decimal, t is the time of the investment, expressed in years, I is the interest earned by P in t years, and S is the amount due at the end of t years. We should not confuse the student by long-winded word-descriptions of how to find the rate when the principal, the interest, and the time are given, or of how to find the principal when the amount, the rate, and the time are given. He should be taught to substitute given quantities in properly selected equations, and then to find the unknowns by the natural algebraic procedure.

In a treatment of the mathematics of investment, simple interest is important not only on its own merits but also as a means of presenting the important terminology of *present value*, *amount*, *accumulation*, and *discount* under a simple guise. In equation 3, we call P the *present value* of the *amount* S. The possession of P to-day is just as desirable as the possession of S at the end of t years. With account taken of the effects of interest, P and S are *equivalent sums* of money due on different dates.

To accumulate a principal P for t years at the rate r means to find the amount S due at the end of t years if P is invested at the rate r. *To discount* an amount S for t years means to find the present value of S on a day t years before it is due. The discount on S is the difference between S and its present value P, or is $(S-P)$.

EXAMPLE 1. (*a*) Discount \$1,150 for $2\frac{1}{2}$ years at the rate $5\frac{1}{2}\%$, simple interest. (*b*) Find the discount on the \$1,150.

Solution. (*a*) We desire to find the present value of the amount $S=\$1,150$, $2\frac{1}{2}$ years before it is due. We use equation 3 with $t=2\frac{1}{2}$, $S=1,150$, and $r=.055$:

$$1,150 = P[1+\frac{5}{2}(.055)].$$

$$P[2+5(.055)] = 2,300.$$

$$2.275\,P = 2.300$$

$$P=\frac{2.300}{2.275}=\$1,010.99.$$

(*b*) Since the present value of \$1,150 is \$1,010.99, the discount is (1,150 — 1,010.99), or \$139.01.

In the preceding example, \$139.01 is the discount on \$1,150, due at the end of $2\frac{1}{2}$ years, and, also, \$139.01 is the interest for $2\frac{1}{2}$ years on the principal \$1,010.99. That is, \$139.01 plays a double rôle.

The last paragraph illustrates an interesting fact. From equation 2, $I=(S-P)$. Hence, the quantity I, which was defined as the *interest* on the principal P, has now been given a second name; I is also the *discount* on the amount S. That is, I plays a double rôle.

Simple Discount. The preceding discussion of the quantity I leads naturally to an important notion. In referring to I as the interest on P, we use the equation $I = Prt$, which expresses I as a certain percentage per year of the principal P. Similarly, in considering I as the discount on the amount S, it is very convenient to express I as a certain percentage per year of the amount S. To do so, we introduce a rate of simple discount, call it d, expressed as a decimal. Then $I = Sdt$. Accordingly, from equation 2

$$P = S - I = S - Sdt = S(1 - dt).$$

Hence, we are led to the following equations of simple discount:

$$I = Sdt; \qquad (4)$$
$$P = S - I; \qquad (5)$$
$$P = S(1 - dt). \qquad (6)$$

Equations 4, 5, and 6 are just as important as equations 1, 2, and 3.

EXAMPLE 2. \$300 is due at the end of 90 days. Find its present value, at 6% simple discount, using 360 days to the year.

Solution. We are given $S = 300$; $d = .06$; $t = \frac{1}{4}$. From equation 4, $I = 300(.06)(\frac{1}{4}) = 4.50$. From equation 5, $P = 300 - 4.50 = \$295.50$.

In current terminology, simple discount is frequently referred to as *simple interest payable in advance.* This is somewhat unfortunate, because simple discount is *discount* and is not *interest*, in the ordinary sense of the word. Nevertheless, if we once recognize that *simple interest in advance means discount*, confusion ceases, provided we admit the independent existence of equations 4, 5, and 6, and do not attempt to make equations 1, 2, and 3 do double duty [1] both for simple interest and for simple discount. All of the usual problems in the discounting of promissory notes, and the other numerous applications of simple discount or (what is the same) of interest payable in advance, can be conveniently handled by use of equations 4, 5, and 6.

EXAMPLE 3. A banker charges 6% simple interest payable in advance. (*a*) What shall I agree to pay to him at the end of 6 months, in order that

[1] The author feels that the frequent failure of textbooks to recognize the independent existence of equations 4, 5, and 6 is an evil which leads to the current lack of clear thinking about discount and interest in advance by the average man.

I may receive $1,000 from him now as the proceeds of my loan? (*b*) What interest rate am I actually paying?

Solution. (*a*) Interest in advance, at the rate 6%, means that simple discount is being charged at the rate 6%. In equations 4, 5, and 6, we are given $P = 1{,}000$, $t = \frac{1}{2}$, and $d = .06$. We use equation 6:

$$1{,}000 = S[1 - \tfrac{1}{2}(.06)];$$
$$.97\,S = 1{,}000;$$
$$S = \frac{1{,}000}{.97} = \$1{,}030.93.$$

(*b*) From Part *a*, I pay $30.93 interest at the end of 6 months in addition to the $1,000 which I received at the beginning of the transaction. From $I = Prt$, with $P = 1{,}000$, $I = 30.93$, and $t = \frac{1}{2}$,

$$30.93 = 1{,}000(r)(\tfrac{1}{2}).$$

On solving, we obtain $r = .0619$. Hence, a rate of 6%, *payable in advance,* on a 6-month loan, is equivalent to a charge of simple interest at the rate 6.19%.

The terminology of present value, amount, accumulation, and discount applies equally well both to simple discount and to simple interest discussions. To discount an amount S, due at the end of t years, under *simple interest* at the rate r, means to find P by use of $S = P(1 + rt)$. To discount an amount S, due at the end of t years, *under simple discount* at the rate d, means to find P by use of $P = S(1 - dt)$. The equations of simple discount are easier to apply than those of simple interest, if we desire to find a principal P when the amount S is given, that is, when we desire *to discount* an amount S. The equations of simple interest are easier to apply than those of simple discount when we desire to find S when P is given.[5]

PART III. COMPOUND INTEREST

Definition of Compound Interest. If, at stated intervals during the term of an investment, the interest due is added to the principal and thereafter earns interest, the sum by which the original principal has increased by the end of the term of the investment is called *compound interest.* At the end of the term, the total amount due, which consists of the original principal plus the compound interest, is called the *compound amount.*

Hereafter, the unqualified word *interest* will always refer to *compound interest.*

[5] For a complete discussion of the relations between simple discount and simple interest, see Chapter 1 of the author's *Mathematics of Investment, Revised,* D. C. Heath and Company, publishers.

We speak of interest being *compounded*, or *converted* into principal. The time between successive conversions of interest into principal is called the *conversion period*, or the *interest period*.

Thus, if the rate is 6%, compounded quarterly, the conversion period is 3 months, and interest is earned at the rate 6% per year during each period, or at the rate 1.5% per conversion period.

EXAMPLE 1. Find the compound amount at the end of 1 year if $100 is invested at the rate 8%, compounded quarterly.

Solution. The rate per conversion period is .02. The original principal is $100. At the end of 3 months, $2.00 interest is due; the new principal is $102. At the end of 6 months, the interest due is 2% of $102, or $2.04, and the new principal is ($102 + $2.04), or $104.04. In this fashion, we find that the amount at the end of 1 year is $108.243. The total compound interest earned in the year is $8.243. The rate at which the original principal increased during the year is $\frac{8.243}{100}$, or 8.243%.

Nominal and Effective Rates. In the usual way of describing a given variety of compound interest, the rate specified is called the *nominal rate*. It is the rate per year at which interest is earned during each conversion period. The *effective rate* is the rate per year at which interest is earned during each year. Thus, in the preceding example, the nominal rate is 8%; the effective rate is 8.243%; the rate per conversion period is 2%. All of these three rates of interest should be kept in mind when considering a given variety of compound interest.

It is emphasized that, in the future, when we refer to a rate of interest, we mean a rate per *year*, except when we otherwise specify a rate *per conversion period*.

The Compound Interest Formula. Let the interest rate per conversion period be r, expressed as a decimal. Let P be the original principal, and let S be the compound amount to which P accumulates by the end of k conversion periods. Then, we shall prove that

$$S = P(1 + r)^k. \tag{1}$$

Proof. The original principal invested is P.
The interest due at the end of 1st period is Pr.
New principal at the end of 1st period is $P + Pr = P(1 + r)$.
Interest due at the end of 2d period is $r[P(1 + r)]$, or $Pr(1 + r)$.
New principal at the end of 2d period is

$$P(1 + r) + P(1 + r)r = P(1 + r)(1 + r), \text{ or } P(1 + r)^2.$$

By the end of each period, the principal on hand at the beginning of the period has been multiplied by $(1+r)$. Hence, by the end of k periods, the original principal P has been multiplied k successive times by $(1+r)$, or by $(1+r)^k$. Therefore, the compound amount at the end of k periods is $P(1+r)^k$, as in equation 1.

The exceedingly elementary discussion leading to equation 1 was not given with the idea that any new facts were being presented. The object of this discussion was to exhibit the convenience of the terminology of *conversion periods,* and *rates per period,* and to show how this terminology leads to a simple general formula for the compound amount.

As in Part II, we call P, in equation 1, the present value of the amount S. To accumulate P for k conversion periods at compound interest, means to find the amount S by use of equation 1.

EXAMPLE 2. Accumulate $300 for 9¼ years at 6%, compounded quarterly.

Solution. The rate per conversion period is $.06 \div 4$, or $r = .015$. The number of conversion periods is $k = 4(9\frac{1}{4}) = 37$. From equation 1,

$$S = 300(1.015)^{37} = 300(1.7348) = \$520.44.$$

In the solution, we obtained the value of $(1.015)^{37}$ from Table I, on page 84.

If Table I were not available, $(1.015)^{37}$ would have to be computed by use of logarithms, or, less conveniently, by use of the binomial theorem. For this reason, one might rashly say that logarithms were essential tools for further work. This statement is unjustified, because fairly complete tables[6] of compound amounts are just as accessible as tables of logarithms. The whole discussion from here on presupposes that some convenient set of interest tables is available. For the illustrations of the present chapter, the accompanying extracts of tables are sufficient.

To discount an amount S for k conversion periods means to find the present value of S on a day k periods before S is due. To find P, the present value, we solve equation 1 for P in terms of S: we find $P = S \div (1+r)^k$, or

$$P = S(1+r)^{-k}. \qquad (2)$$

Although equations 1 and 2 are equivalent equations, we refer to them as the fundamental *equations* of compound interest.

EXAMPLE 3. Discount $50 for 4½ years at 4%, compounded semiannually.

Solution. In equation 2, we have $S = \$50$, $r = .02$, and $k = 2(4\frac{1}{2}) = 9$.

$$P = 50(1.02)^{-9} = 50(.83676) = \$41.84.$$

The discount on the $50 is $(50 - 41.84)$, or $8.16.

[6] See, for instance, the author's *Tables from the Mathematics of Investment,* D. C. Heath and Company, publishers.

Comment on the Fundamental Formulas. Recall that, in equations 1 and 2, the unit of time is the *conversion period*, and not *one year*; the symbol r is the rate per conversion period; k is the time, expressed in conversion periods. This feature results in a worthwhile simplification. If we had let n be the number of years for which P is invested, m be the number of times that interest is converted in each year, and j be the nominal rate, then we would have found that

$$S = P\left(1 + \frac{j}{m}\right)^{mn}. \tag{3}$$

The greater complication of (3), as compared with (1), is the reason why we used the conversion period instead of the year as the fundamental unit of time for the description of data. We do not employ equation 3, and recommend its complete submersion.

Compound Interest for Fractional Periods. The fundamental definition of compound interest gives no meaning to the notion unless the time of the investment is an integral number of conversion periods. That is, up to the present, we have assumed that k, in equation 1, is an integer. If the time of the investment is not a whole number of conversion periods, it is customary in practice to define the compound amount to be the result obtained as follows:

(1) *Find the compound amount at the end of the last whole conversion period contained in the given time.*

(2) *Accumulate the resulting amount for the remainder of the time at simple interest at the given nominal rate.*

EXAMPLE 4. Find the amount at the end of 2 years and 7 months, if \$1,000 is invested at 8%, compounded quarterly.

Solution. The last interest date in the term of the investment is at the end of $2\frac{1}{2}$ years. The amount at the end of $2\frac{1}{2}$ years is, by equation 1,

$$\$1,000(1.02)^{10} = 1,000(1.2190) = \$1,219.00.$$

The remaining time is 1 month. Hence, we accumulate the new principal, \$1,219, for 1 month at simple interest, at the rate 8%. We use $I = Prt$ with $P = \$1,219$, $r = .08$, and $t = \frac{1}{12}$:

$$I = 1,219(.08)(\tfrac{1}{12}) = \$8.13.$$

The amount at the end of 2 years and 7 months is (\$1,219 + \$8.13), or \$1,227.13.

In place of the definition of the compound amount for a fractional period which we have just used, it is customary to agree, in theoretical work, that, *even when k is not an integer*, the present value P and the amount S shall be related by equation 1. For contrast, the definition used in Example 4 may be called the *practical* definition of the amount for a fractional period, and the second definition may be called the *theoretical* definition. The amount given by the practical definition differs only slightly, in the usual problem, from the amount given by the theoretical definition.

Unknown Rates and Times are conveniently found by simple interpolation in compound interest tables. The process of interpolation referred to is the same as that employed in the use of tables of logarithms or tables of the trigonometric functions.

EXAMPLE 5. If interest is compounded semiannually, find the nominal rate at which \$1,000 accumulates to \$1,200 in $5\frac{1}{2}$ years.

Solution. Let r be the unknown rate per conversion period. We have $P = \$1{,}000$, $S = \$1{,}200$, and $k = 11$. From equation 1,

$$1{,}200 = 1{,}000(1 + r)^{11},$$
$$(1 + r)^{11} = 1.200.$$

In the row of Table I for $k = 11$, we find $(1.015)^{11} = 1.1779$, which is *less* than 1.200, and $(1.02)^{11} = 1.2434$, which is *more* than 1.200. Hence, r is between .015 and .02. In finding r by interpolation, we assume that r is the same proportion of the way from .015 to .02 as 1.200 is of the way from 1.1779 to 1.2434.

i	$(1 + i)^{11}$
.015	1.1779
$r = ?$	1.2000
.02	1.2434

$$1.2434 - 1.1779 = .0655;$$
$$1.2000 - 1.1779 = .0221.$$

Hence, r is $\frac{.0221}{.0655}$ of the way from .015 to .02. Since $(.02 - .015) = .005$,

$$r = .015 + \frac{221}{655}(.005) = .015 + .0017 = .0167.$$

The nominal rate is $2r$, or .0334, or approximately 3.3%. This final result is almost certainly accurate to tenths of a per cent. A second approximation could be found by interpolation, by use of considerable computation,[1] but the present result would satisfy most practical needs.

EXAMPLE 6. How long will it take for \$5,250 to accumulate to \$7,375 if the money is invested at 6%, compounded quarterly?

Solution. Let k be the necessary number of conversion periods. From equation 1,

$$7{,}375 = 5{,}250(1.015)^k;$$
$$(1.015)^k = \frac{7{,}375}{5{,}250} = 1.4048.$$

By interpolation in the $1\frac{1}{2}\%$ column of Table I, we find that $k = 22.83$, periods of 3 months. The time required is $\frac{1}{4}(22.83)$ years, or 5.71 years.

[1] See Comment 1, p. 34, in the author's *Mathematics of Investment*.

There is ample justification, on the grounds of accuracy and simplicity, for using a certain logarithmic method in Example 5. However, in the analogous and more important type of problem relating to annuities, this logarithmic method is not applicable. For this reason, we adopt the interpolation method as our standard one for determining unknown rates.

In Example 6, we solved an exponential equation in k by use of interpolation. It can be proved [8] that the solution obtained by interpolation is an *exact* solution of the problem, subject to the natural limitations of the tables employed, if the practical definition of the previous section is adopted for the compound amount for a fractional period. On the other hand, a solution of the exponential equation in Example 6 by use of the customary logarithmic method gives an inaccurate result, according to practice as exemplified by the practical definition of Section 5. This is an interesting point, in view of current impressions in regard to the approximate nature of results obtained by use of interpolation methods.

Equations of Value. To compare two sets of financial obligations involving sums of money due on various dates, we must first reduce all sums involved to equivalent sums due on some common comparison date. An *equation of value* is an equation stating that the sum of the values, on a certain comparison date, of one set of obligations equals the sum of the values on this date of another set. Equations of value are powerful tools for solving problems throughout the mathematics of investment.

EXAMPLE 7. W owes Y (*a*) \$100 due at the end of 10 years, and (*b*) \$200 due at the end of 5 years with accumulated interest at the rate 3%, compounded semiannually. W wishes to pay in full by making two equal payments at the ends of the 3d and the 6th years. If money is now considered worth 4%, compounded semiannually to the creditor Y, find the size of W's equal payments.

Solution. Let $\$x$ be the payment. W wishes to replace his old obligations by two new ones. By equation 1, Part III, obligation b requires the payment of $200(1.015)^{10}$ at the end of 5 years.

Old Obligations	*New Obligations*
(*a*) \$100 due at the end of 10 yr.	$\$x$ due in 3 yr.
(*b*) $\$200(1.015)^{10}$ due at the end of 5 yr.	$\$x$ due in 6 yr.

We shall use the end of 5 years as a comparison date. In the following equation of value, the left member is the sum of the equivalent values of

[8] See W. L. Hart, *American Mathematical Monthly*, Vol. XXXVI (1929), p. 379.

the old obligations at the end of 5 years, and the right member is the sum of the equivalent values of the new obligations at the end of 5 years. In writing the left member, we *discount* (*a*) for 5 years, and take (*b*) unchanged, because (*b*) is due at the end of 5 years. In writing the right member, we *accumulate* the first $x for 2 years, and *discount* the second $x for 1 year. In discounting, or accumulating, we use equation 2, or equation 1, as the case may be.

$$100(1.02)^{-10} + 200(1.015)^{10} = x(1.02)^{4} + x(1.02)^{-2}.$$
$$100(.8203) + 200(1.1605) = x(1.0824) + x(.9612).$$
$$314.13 = x(1.0824 + .9612) = 2.0436x.$$
$$x = \frac{314.13}{2.0436} = \$153.72.$$

Part IV. Annuities Certain

Annuities. An annuity is a sequence of equal periodic payments. An *annuity certain* is one whose payments extend over a fixed period of time; a *contingent* annuity is one whose payments extend over a period of time the length of which depends on events whose dates of occurrence cannot be accurately foretold.

Thus, a sequence of equal payments made in purchasing a house on the installment plan forms an annuity certain. The premiums on a life insurance policy form a contingent annuity because the premiums cease at the death of the insured person.

We shall deal only with annuities certain, and, in this discussion, the qualifying word *certain* will be omitted. The sum of the payments of an annuity which are made in one year is called the *annual rent*. The time between successive payments is called the *payment interval*. The time between the beginning of the first payment interval and the end of the last one is called the *term* of the annuity. Unless otherwise stated, the payments of an annuity are due at the ends of the payment intervals; the first payment is due at the end of the first interval, and the last payment is due at the end of the term of the annuity.

Thus, in the case of an annuity of $150 per month for 15 years, the payment interval is one month, the annual rent is $1800, and the term is 15 years.

Present Value and Amount of an Annuity. Under a specified rate of interest, the *present value* of an annuity is the sum of the present values of all the payments of the annuity. The *amount* of an annuity is the sum of the compound amounts which would be on hand at the end of the term of the annuity if all payments should accumulate until then from the dates on which they are due.

Illustration. Consider an annuity of \$100, payable annually for 5 years. Suppose that interest is at the rate 4%, compounded annually. We obtain the present value A of this annuity by adding the 2d column of the following table, and the amount S by adding the 3d column. The values of the various powers of (1.04) were taken from a table not included with this chapter.

Payment of \$100 Due at End of	Present Value of Payment	Compound Amount at End of Term if Payment Is Left to Accumulate at Interest
1 year	$100(1.04)^{-1} = 96.154$	$100(1.04)^4 = 116.986$
2 years	$100(1.04)^{-2} = 92.456$	$100(1.04)^3 = 112.486$
3 years	$100(1.04)^{-3} = 88.900$	$100(1.04)^2 = 108.160$
4 years	$100(1.04)^{-4} = 85.480$	$100(1.04) = 104.000$
5 years	$100(1.04)^{-5} = 82.193$	$100 = 100.000$
	(add) $A = \$445.183$	(add) $S = \$541.632$

Annuity Formulas. In applications, the payment interval of the annuity involved is usually found to be the same as the conversion period of the interest rate. This is the only case which we shall treat.[9]

Consider an annuity which pays \$1 at the end of each interest period for n interest periods. Let ($a_{\overline{n|}}$ at i) represent the present value, and ($s_{\overline{n|}}$ at i) represent the amount of this annuity when the interest rate per conversion period is i.

To determine a formula for $a_{\overline{n|}}$. The first \$1 payment is due at the end of one interest period, and has the present value $(1+i)^{-1}$. The present value of the second payment is $(1+i)^{-2}$; etc., the present value of the next to the last payment, due at the end of $(n-1)$ periods, is $(1+i)^{-(n-1)}$. The present value of the last payment is $(1+i)^{-n}$. The present value of the annuity is the sum of these present values, or

$$a_{\overline{n|}} = (1+i)^{-n} + (1+i)^{-(n-1)} + \ldots + (1+i)^{-2} + (1+i)^{-1};$$
$$a_{\overline{n|}} = (1+i)^{-n} + (1+i)^{-n+1} + \ldots + (1+i)^{-2} + (1+i)^{-1}. \quad (1)$$

On the right side of equation 1, we meet a geometric progression, with the common ratio equal to $(1+i)$. The formula for the sum of a geometric progression is $(rl - a) \div (r-1)$, where r is the ratio, l is the last term, and a is the first term. In equation 1, we

[9] For a general treatment of all cases which arise in the applications of annuities, see author's *Mathematics of Investment*, Chapter IV.

have $r = (1 + i)$, $l = (1 + i)^{-1}$, and $a = (1 + i)^{-n}$. Hence, we find that

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}. \tag{2}$$

To find a formula for $s_{\overline{n}|}$. We recall that if we accumulate a present value we obtain the corresponding amount. Hence, since $a_{\overline{n}|}$ is the present value of the payments of the annuity, and $s_{\overline{n}|}$ is the sum of the amounts of the same payments at the end of the n periods which form the term of the annuity, we can obtain $s_{\overline{n}|}$ by accumulating $a_{\overline{n}|}$ for n periods. Therefore, by formula 1 of Part III, with $P = a_{\overline{n}|}$, $r = i$, and $k = n$,

$$s_{\overline{n}|} = a_{\overline{n}|}(1 + i)^n = \left(\frac{1 - (1 + i)^{-n}}{i}\right)(1 + i)^n.$$

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}. \tag{3}$$

Now, consider an annuity problem in which R is the periodic payment, i is the interest rate per conversion period, and n is the number of periods in the term. Let A represent the present value, and S the amount of the annuity. The present value A will be R times $a_{\overline{n}|}$, which is the present value when each payment is \$1 instead of \$$R$; and, similarly, the amount S will be R times $s_{\overline{n}|}$. Hence,

$$A = Ra_{\overline{n}|}; \tag{4}$$

$$S = Rs_{\overline{n}|}. \tag{5}$$

The relative complication of formulas 2 and 3 might lead one to think that any application of the annuity formulas would be tedious. However, the existence of excellent annuity tables allows us to completely submerge equations 2 and 3, and to work entirely with the simple equations 4 and 5, except in unusual problems which need not be considered in an introduction to the subject.

EXAMPLE 1. In purchasing a house, a man agrees to pay \$6,000 cash, and \$1,000 at the end of each 6 months for 6 years. If money is worth 4%, compounded semiannually, find the equivalent cash price of the house.

Solution. The \$1,000 payments form an annuity whose term is 6 years; the interest rate per conversion period is .02; the term is 12 periods long. The equivalent cash value of the annuity is its present value A; from equation 4,

$$A = 1{,}000(a_{\overline{12}|} \text{ at } .02) = 1{,}000(10.57534). \quad \text{(From Table IV)}$$

Hence, $A = \$10{,}575.30$. The cash value of the house is (\$6,000 + A), or approximately \$16,575.

EXAMPLE 2. If you deposit \$50 at the end of each 6 months in a bank which credits interest semiannually at the rate 4%, how much will be to your credit at the end of the 20th year?

Solution. The amount to your credit is the amount S of the annuity formed by the deposits. We use equation 5 with $R = \$50$, $n = 40$, and $i = .02$.

$$S = 50(s_{\overline{40}|} \text{ at } .02) = 50(60.4020) = \$3{,}020.10;$$

in the solution, we used Table III.

EXAMPLE 3. I purchase a house worth \$12,000 cash. I pay \$2,000 cash. I also agree to make equal payments at the end of each 3 months for 9 years to discharge the balance, principal and interest included. If interest is at the rate 6%, compounded quarterly, what must I pay at the end of each 3 months?

Solution. After paying \$2,000 cash, the balance is \$10,000. This \$10,000 is the present value of the annuity which I shall pay, quarterly, in discharging my debt. Let R be the unknown quarterly payment. Then, we use equation 4 with $A = \$10{,}000$, R unknown, $n = 36$, and $i = .015$:

$$10{,}000 = R(a_{\overline{36}|} \text{ at } .015).$$

$$R = \frac{10{,}000}{(a_{\overline{36}|} \text{ at } .015)} = \frac{10{,}000}{27.6607} = \$361.52 \qquad \text{(Table IV)}$$

Amortization of a Debt. A debt, whose present value is A, is said to be *amortized* under a given rate of interest, if all liabilities as to principal and interest are discharged by a sequence of periodic payments. When the payments are equal, as is usually the case, they form an annuity whose present value must equal A, the original liability. In the preceding Example 3, we determined the quarterly payment which would *amortize* a debt of \$10,000 with interest at 6%, compounded quarterly, in 9 years.

EXAMPLE 4. A farm is worth \$25,000 cash. A buyer will pay \$12,000 cash, and he will amortize the balance by payments of \$1,000 at the end of each 3 months for 3 years and 9 months. At what interest rate, payable quarterly, is the transaction being executed?

Solution. The balance due, after the cash payment, is \$13,000. Let i be the unknown interest rate, per conversion period (of 3 months). Then, \$13,000 must equal the present value, at the rate i, of an annuity of \$1,000 payable quarterly for 3 years and 9 months. We use formula 4, with $n = 15$, $R = \$1{,}000$, $A = \$13{,}000$, and i unknown:

$$13{,}000 = 1{,}000(a_{\overline{15}|} \text{ at } i); \qquad (a_{\overline{15}|} \text{ at } i) = 13.$$

We solve the last equation by interpolation in Table IV. We see from the table that $(a_{\overline{15}|}$ at $1\frac{1}{2}\%) = 13.343$, and $(a_{\overline{15}|}$ at $2\%) = 12.849$. Hence, i is between $1\frac{1}{2}\%$ and 2%. By a solution like that of Example 5 of Part III, we find that $i = .0185$; since this is the rate per 3 months, the nominal rate is $4i = .0740$, or 7.4%, approximately.

EXAMPLE 5. I purchase a building worth \$20,000 cash, and agree to make equal payments at the beginning of each 3 months for 8 years, to completely discharge all of my liability as to principal and interest. If interest is at the rate 6%, compounded quarterly, find my quarterly payment.

Solution. Let R be the unknown quarterly payment. If, for the moment, we disregard the 1st payment, the remaining payments form an annuity whose term commences on the day of purchase; the term of this annuity is 7¾ years, because the last payment occurs at the beginning of the last period of 3 months in the 8 years. Hence, the present value of all of the payments, omitting the first one, is $Ra_{\overline{31}|}$. The cash value of the house equals the 1st payment R, which is cash, plus the present value of the remaining payments. Therefore,

$$20{,}000 = R + R(a_{\overline{31}|} \text{ at } .015).$$
$$20{,}000 = R(1 + a_{\overline{31}|}).$$
$$20{,}000 = R(1 + 24.6461). \qquad \text{(Using Table IV)}$$
$$R = \frac{20{,}000}{25.6461} = \$779.85.$$

In Example 5, the quarterly payments furnish an illustration of an *annuity due*. A sequence of equal periodic payments is said to form an *annuity due*, as judged from a certain date, if the payments occur at the *beginnings* of the payment intervals. The first payment of an annuity due is payable at the beginning of its term, and the last payment is payable one period before the end of the term. In Example 5, we could say that the cash value of the building is the present value of the annuity due formed by the payments. In order to determine the present values, or amounts of annuities due, we do not have to develop new formulas; we solve such problems by suitable manipulation[10] of the fundamental formulas 3 and 4.

Determination of a Final Amortization Payment. If a debt of specified size is to be amortized, at a given rate of interest, by periodic payments of specified size, we meet the problem of determining when these payments should cease and of finding the final irregular payment which will close the transaction. The final payment is almost certainly irregular, as will be easily inferred from the solution of the next example.

EXAMPLE 6. A man borrows \$10,000 with the agreement that money is worth 8%, compounded quarterly. To discharge his principal and interest

[10] For a complete discussion of annuities due, see the author's *Mathematics of Investment*, p. 65.

obligations, he will pay \$500 at the end of each 3 months as long as necessary, and he will pay whatever final installment is necessary to close the transaction 3 months after the last \$500 payment which is required. Find (*a*) how many \$500 payments will be required; (*b*) the final smaller payment.

Solution. (*a*) If n payments of \$500 each were exactly sufficient to discharge the debt, then the present value of an annuity of \$500 paid quarterly for n periods would equal \$10,000; that is, by equation 4 with $i = .02$,

$$10{,}000 = 500(a_{\overline{n}|} \text{ at } .02).$$

$$(a_{\overline{n}|} \text{ at } .02) = 20. \qquad (6)$$

From Table IV, we see that $(a_{\overline{25}|}$ at $.02) = 19.5$, which is less than 20. Hence, 25 payments of \$500 each would *not be sufficient* to discharge the debt. Also, from Table IV, $(a_{\overline{26}|}$ at $.02) = 20.12$; hence, since this is *more* than 20, it follows that 26 payments of \$500 each would be *more* than is required to discharge the debt. These facts show that the debtor should pay 25 installments of \$500 each, and some amount W, *less than* \$500, to close the transaction at the end of the 26th period. The last \$500 is due at the end of 25 periods, or $6\frac{1}{4}$ years.

(*b*) To determine the unknown final payment, W, we proceed as follows. First, we solve equation 6 for n by interpolation in Table IV. We find that

$$n = 25 + \frac{4{,}765}{5{,}975} = 25.7975.$$

Then, Theorem I, which follows, states that the final payment is $500(.7975)$, or \$398.75.

The solution of the last example serves to illustrate the following theorem.

THEOREM I. When a debt A is discharged, principal and interest included, by payments of R at the end of each interest period for as long as necessary, with an additional smaller payment one period after the last installment R, then the date and size of the final small payment can be found as follows:

1. *Solve $A = R(a_{\overline{n}|}$ at $i)$ for n by interpolation in Table IV.*

2. *If the solution obtained is $n = k + f$, where k is a positive integer and f is a positive number less than 1, then the final payment is due at the end of $(k + 1)$ interest periods, and*

$$\textit{final payment} = fR. \qquad (7)$$

The essential part of the statement of Theorem I is in equation 7, whose proof is beyond the scope of the present chapter.[11] Equation 7 was used in the solution of part b of Example 6. In

[11] For a proof, see W. L. Hart, *American Mathematical Monthly*, Vol. XXXVI (1929), p. 379.

equation 7 we have another interesting illustration of an exact result, obtained through an interpolation process. A solution of the equation $A = Ra_{\overline{n}|}$ by a logarithmic method, employing the explicit formula 2, would give a decimal f_1 nearly the same as the f of equation 7, which was obtained by interpolation. But, the number f_1 would not have the property which is possessed by f, and is stated in equation 7.

Sinking Funds. A sinking fund is a fund accumulated to pay an obligation falling due at some future date. If a fund is accumulated by investing equal periodic deposits, the amount in the fund at any time is the *amount of the annuity* formed by the deposits.

Suppose that $A is borrowed, with the agreement that interest is payable as due, and that the principal shall be paid in one installment at the end of h years. If the debtor provides for the future payment of $A by the creation of a sinking fund, he is said to discharge his debt by the *sinking fund method*. Under this method, we shall assume that the debtor makes equal deposits in his sinking fund on the same days that he pays interest to his creditor. The debtor's periodic expense on account of the debt is the sum of the following items:

(a) Payment to the creditor of the interest due on $A.

(b) Deposit for the sinking fund, which is to accumulate to $A by the end of the term of the transaction.

EXAMPLE 7. A man borrows $10,000, and agrees to pay interest semiannually at the rate 6%, and to pay the principal in one installment at the end of $2\frac{1}{2}$ years. If the debtor uses the sinking fund method, and invests his fund at 4%, compounded semiannually, find the semiannual expense of the debtor on account of the debt.

Solution. Semiannual interest at 6%, which is payable to the creditor, is .03(10,000), or $300.

Let R be the semiannual deposit in the sinking fund. These deposits form an annuity whose term is $2\frac{1}{2}$ years, and whose amount must be $10,000, because the fund is to provide for the repayment of the debt at the end of $2\frac{1}{2}$ years. Since the fund accumulates at 4%, compounded semiannually, we use equation 5 with $i = .02$, $n = 5$, $S = 10{,}000$, and R unknown:

$$10{,}000 = Rs_{\overline{5}|.02} = R(5.2040).$$

$$R = \frac{10{,}000}{5.2040} = \$1{,}921.6.$$

Hence, the semiannual expense of the debtor is (300 + 1,921.6), or approximately $2,222.

Bonds. A bond is a written contract to pay a definite *redemption price* $\$C$ on a specified redemption date, and to pay equal *dividends* $\$D$ periodically until after the redemption date. The principal $\$F$ mentioned in the face of the bond is called the *face value*, or the *par value*. A bond is said to be redeemed at par if $C = F$ (as is usually the case), and at a premium if C is greater than F. The interest rate named in a bond is called the *dividend rate*, or the bond rate. The dividend D is described in a bond by saying that it is the interest, semiannual or otherwise, on the par value F at the dividend rate. The following paragraph illustrates the essential paragraph in a bond:

> The Kansas Improvement Corporation acknowledges itself to owe and, for value received, promises to pay the bearer Five Hundred Dollars on January 1, 1936, with interest on the said sum from and after January 1, 1925, at the rate 6% per annum, payable semiannually, until the said principal sum is paid. Furthermore, an additional 10% of the said principal shall be paid to the bearer on the date of redemption.

For the bond of the preceding paragraph, $F = \$500$, $C = \$550$, and the semiannual dividend $D = \$15$, which is semiannual interest at 6% on \$500. A bond is named after its face F and dividend rate, so that the preceding paragraph is an extract from a \$500, 6% bond. Corresponding to each dividend, there would usually be attached to the bond an individual written contract to pay $\$D$ on the proper date.

When an investor purchases a bond, the interest rate which his investment yields is computed under the assumption that he will hold the bond until it is redeemed.

When a bond is sold on a dividend date, the seller takes the dividend which is due. The purchaser receives the future dividends, which form an annuity whose first payment is due at the end of one dividend interval, and whose last payment is due on the redemption date. If an investor desires a specified yield on purchasing the bond, the price $\$P$ which he should be willing to pay is given by the following equation, where present values are computed at the investment rate which the buyer has specified:

$$P = (\textit{present value of } \$C, \textit{ due on the redemption date}) + (\textit{present value of the annuity formed by the dividends}). \quad (8)$$

EXAMPLE 8. A \$100, 6% bond, with dividends payable semiannually, will be redeemed at par at the end of 15 years. Find the price of the bond at which a purchaser will obtain a return of 4%, compounded semiannually, on his investment.

Solution. The semiannual dividend is \$3. The redemption value is \$100, due at the end of 15 years. The term of the dividend annuity is 15 years, or 30 interest periods. By use of equation 8,

$$P = 100(1.02)^{-30} + 3(a_{\overline{30}|} \text{ at } .02) = \$122.40. \qquad (9)$$

In equation 9, we used Table II and also Table IV.

Determination of Bond Yields. A purchaser of a bond cannot, as a rule, dictate the price which he will pay, or the interest rate which the bond shall yield, although, of course, he may withhold his purchase until the conditions satisfy him. Essentially, a purchaser of a bond must pay a price which is specified by the law of supply and demand, as exemplified in the prices on the bond market. Hence, we are led to the problem of determining the investment yield which is given by the purchase of a bond at a given price. This problem is the converse of that treated in the preceding example, where a yield was given and we determined the price. A solution of this converse problem is too lengthy for the present discussion, although the solution presents no serious difficulty. If an annuity table, like Table IV, is available, the problem can be solved by interpolation, after a suitable arrangement of preliminary details. The solution is particularly easy if a bond table is available giving the prices of bonds at various yields and for various times to the redemption date.

An approximate solution of the problem of determining a bond yield can be obtained by the method of the next two examples. Example 9 illustrates the case of a bond bought at a discount, that is, bought for less than the redemption payment. Example 10 illustrates the case of a bond bought at a premium, that is, bought for more than the redemption payment. It is interesting to note that the method[1] of these examples involves mere arithmetic.

EXAMPLE 9. A \$1,000, 6% bond pays dividends annually and is redeemable at par at the end of 10 years. If the bond is bought for \$850, determine an approximation to the yield which the investor obtains.

Solution. Each dividend is \$60. The investor pays \$850 for the bond, and receives \$1,000 at the end of 10 years, besides the dividends. This difference (1,000 − 850), or \$150, represents the accumulated value, at the end of 10 years, of interest on the investment besides that which the dividends provided. During the 10 years, the investor should think of the value of

[1] For a theoretical justification of the method, see author's *Mathematics of Investment*, p. 248.

his bond as increasing from \$850 to \$1,000. If the \$150 increase were spread uniformly [13] over the 10 years, the increase per year would be $\frac{1}{10}(150)$, or \$15. This \$15, together with the annual dividend of \$60, or \$75 in all, is approximately the annual interest return from the investment. The average value of the investment is $\frac{1}{2}(850 + 1,000)$, or \$925, because the value of the bond increases from \$850 to \$1,000. The interest rate obtained by the investor is approximately the rate at which \$75 is the interest for one year on \$925; this rate is $\frac{75}{925}$, or .081. That is, the bond yields approximately 8.1% per year.

It can be shown that the method of the preceding example furnishes a result which is usually in error by not more than .2%, as a very extreme limit, in the customary type of problem.

EXAMPLE 10. If the bond of Example 9 is bought for \$1,200, find an approximation to the yield obtained by the investor.

Solution. Each dividend is \$60. The investor pays \$1,200, but receives only \$1,000 at the end of 10 years, besides the dividends. It is *not correct* to say that the difference (1,200 − 1,000), or \$200, represents a loss of principal. We should say that this \$200 decrease represents a return of principal which has resulted through the dividend payments. Therefore, each dividend consists only partly of interest; part of each dividend is a return of principal. If the \$200 return of principal were spread uniformly [14] over the 10 years, the return per year would be $\frac{1}{10}(200)$, or \$20. Hence, in each dividend of \$60, there is a return of about \$20 of principal; the balance, or \$40, is interest. The value of the bond decreases from \$1,200 to \$1,000 during the 10 years; the average value is $\frac{1}{2}(1,200 + 1,000)$, or \$1,100. The interest rate which the investor obtains is approximately equal to the rate at which \$40 is interest for one year on \$1,100; this rate is $\frac{40}{1,100}$, or .0364. The investor obtains approximately 3.6% on his investment.

PART V. CONCLUSION

The discussion of Parts II, III, and IV was built around an extremely small number of equations, three for simple interest, three for simple discount, two for compound interest, and two for annuities. In each problem considered, the solution started directly from one of these fundamental equations. Attention is called to the relative simplicity of such a method, in contrast to one where we would employ the auxiliary set of formulas obtained by solving, in turn, in each equation, for each symbol in terms of the others present. This second method, as applied to the equation

[13] Actually, the increase is not spread uniformly over the life of the bond.
[14] Actually, the return of principal is not spread uniformly over the time.

$S = P(1 + r)^k$, would demand that we solve the equation for r, and for k, as a preliminary to our discussion of compound interest; the solutions would be

$$k = \frac{\log S - \log P}{\log (1 + r)}; \quad r = \sqrt[k]{\frac{S}{P}} - 1 \qquad (1)$$

The addition of equations 1 to the two simple formulas which we have used in discussing compound interest would give a rather complicated mathematical background to an otherwise simple subject. In Parts II, III, and IV, the restriction of our formulas to the few simple equations which we employed gives a mathematical background which is easy to remember and convenient to apply.

The methods employed in this chapter were absolutely dependent on the use of previously prepared tables of the values of the fundamental expressions involved. Failure to use such tables to the fullest extent would result in an artificial presentation having few contacts with actual practice in those fields where the subject finds use.

The author believes that a large amount of the subject matter of this chapter merits consideration as a part of any advanced course in arithmetic given in the senior high school.

TABLE I

$(1+i)^k$

k	$1\frac{1}{2}\%$	2%
1	1.0150	1.0200
2	1.0302	1.0404
3	1.0457	1.0612
4	1.0614	1.0824
5	1.0773	1.1041
6	1.0934	1.1262
7	1.1098	1.1487
8	1.1265	1.1717
9	1.1434	1.1951
10	1.1605	1.2190
11	1.1779	1.2434
12	1.1956	1.2682
13	1.2136	1.2936
14	1.2318	1.3195
15	1.2502	1.3459
16	1.2690	1.3728
17	1.2880	1.4002
18	1.3073	1.4282
19	1.3270	1.4568
20	1.3469	1.4859
21	1.3671	1.5157
22	1.3876	1.5460
23	1.4084	1.5769
24	1.4295	1.6084
25	1.4509	1.6406
26	1.4727	1.6734
27	1.4948	1.7069
28	1.5172	1.7410
29	1.5400	1.7758
30	1.5631	1.8114
31	1.5865	1.8476
32	1.6103	1.8845
33	1.6345	1.9222
34	1.6590	1.9607
35	1.6839	1.9999
36	1.7091	2.0399
37	1.7348	2.0807
38	1.7608	2.1223
39	1.7872	2.1647
40	1.8140	2.2080

TABLE II

$(1+i)^{-k}$

k	$1\frac{1}{2}\%$	2%
1	.985 22	.980 39
2	.970 66	.961 17
3	.956 32	.942 32
4	.942 18	.923 85
5	.928 26	.905 73
6	.914 54	.887 97
7	.901 03	.870 56
8	.887 71	.853 49
9	.874 59	.836 76
10	.861 67	.820 35
11	.848 93	.804 26
12	.836 39	.788 49
13	.824 03	.773 03
14	.811 85	.757 88
15	.799 85	.743 01
16	.788 03	.728 45
17	.776 39	.714 16
18	.764 91	.700 16
19	.753 61	.686 43
20	.742 47	.672 97
21	.731 50	.659 78
22	.720 69	.646 84
23	.710 04	.634 16
24	.699 54	.621 72
25	.689 21	.609 53
26	.679 02	.597 58
27	.668 99	.585 86
28	.659 10	.574 37
29	.649 36	.563 11
30	.639 76	.552 07
31	.630 31	.541 25
32	.620 99	.530 63
33	.611 82	.520 23
34	.602 77	.510 03
35	.593 87	.500 03
36	.585 09	.490 22
37	.576 44	.480 61
38	.567 92	.471 19
39	.559 53	.461 95
40	.551 26	.452 89

TABLE III

$s_{\overline{n}|}$

n	1½%	2%
1	1.0000	1.0000
2	2.0150	2.0200
3	3.0452	3.0604
4	4.0909	4.1216
5	5.1523	5.2040
6	6.2296	6.3081
7	7.3230	7.4343
8	8.4328	8.5830
9	9.5593	9.7546
10	10.7027	10.9497
11	11.8633	12.1687
12	13.0412	13.4121
13	14.2368	14.6803
14	15.4504	15.9739
15	16.6821	17.2934
16	17.9324	18.6393
17	19.2014	20.0121
18	20.4894	21.4123
19	21.7967	22.8406
20	23.1237	24.2974
21	24.4705	25.7833
22	25.8376	27.2990
23	27.2251	28.8450
24	28.6335	30.4219
25	30.0630	32.0303
26	31.5140	33.6709
27	32.9867	35.3443
28	34.4815	37.0512
29	35.9987	38.7922
30	37.5387	40.5681
31	39.1018	42.3794
32	40.6883	44.2270
33	42.2986	46.1116
34	43.9331	48.0338
35	45.5921	49.9945
36	47.2760	51.9944
37	48.9851	54.0343
38	50.7199	56.1149
39	52.4807	58.2372
40	54.2679	60.4020

TABLE IV

$a_{\overline{n}|}$

n	1½%	2%
1	.9852	.9804
2	1.9559	1.9416
3	2.9122	2.8839
4	3.8544	3.8077
5	4.7826	4.7135
6	5.6972	5.6014
7	6.5982	6.4720
8	7.4859	7.3255
9	8.3605	8.1622
10	9.2222	8.9826
11	10.0711	9.7868
12	10.9075	10.5753
13	11.7315	11.3484
14	12.5434	12.1062
15	13.3432	12.8493
16	14.1313	13.5777
17	14.9076	14.2919
18	15.6726	14.9920
19	16.4262	15.6785
20	17.1686	16.3514
21	17.9001	17.0112
22	18.6208	17.6580
23	19.3309	18.2922
24	20.0304	18.9139
25	20.7196	19.5235
26	21.3986	20.1210
27	22.0676	20.7069
28	22.7267	21.2813
29	23.3761	21.8444
30	24.0158	22.3965
31	24.6461	22.9377
32	25.2671	23.4683
33	25.8790	23.9886
34	26.4817	24.4986
35	27.0756	24.9986
36	27.6607	25.4888
37	28.2371	25.9695
38	28.8051	26.4406
39	29.3646	26.9026
40	29.9158	27.3555

MATHEMATICS IN AGRICULTURE *

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INTRODUCTION

Value of Mathematics in Agriculture. There are still many people who question the value or the need of any appreciable amount of mathematical training in the field of modern agriculture. On this account, at the outset, it will be well to consider the present structure and relationships of the agricultural industry.

The Business of Farming. The modern farmer, if he is to be successful, must be a business man in the fullest sense. He must have in his mental equipment a larger and broader knowledge of scientific, economic, and financial principles than that required in almost any other line of business in the world to-day in order that he may plan, direct, and carry out the operation of his farm intelligently and with profit. He must be able to plan the size and arrangement of buildings on the farmstead, to subdivide his farm into fields of the proper shape, size, and arrangement for economical and balanced operation, to plan suitable rotations, to inaugurate and carry out systems of fertilization and pest control, to build a farm calendar, and to schedule the amount and distribution of all man and horse labor, and mechanical power. He must know something of the relative efficiency of methods and equipment as well as of the principles of depreciation and replacement of both machinery and livestock. He must balance his crop production to meet the needs of his livestock, and he must plan his feeding rations and select and breed new strains in both crops and livestock to insure best results in production. He must be able to estimate the amount of paint required for a building, or the best shape or size for a given field to be fenced with a known amount of fencing. He must know the capacities of men and machinery, of bins and

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buildings. He must have a knowledge of financing that includes investments, income, loans, capitalization, and amortization. Above all he must know markets, be able to build comparative tables and charts of yields, costs of production, and prices through a series of years, and from these to forecast, with reasonable assurance, the prospects for the current and the coming year for the various crops, as a guide to his general plan of farming for that year.

This is a large contract and it is plainly to be seen that it involves a practical knowledge of mathematics far beyond the scope of the mere traditional third R of our grandfather's day. It includes mensuration, proportion, rates and the compound interest idea in annuities and amortization, graphs and their interpretation, frequency distribution, and the application of mathematics to the elementary principles and practice of surveying.

Related Industry and Commerce. Agriculture is, and will continue to be, the basic industry in this country. Therefore, whether we recognize it or not, a very large part of the general industrial and commercial activity of the country is attuned to harmonize with the agricultural field. Millers of flour, bakers, meat packers—in a word, all processors of human and animal foodstuffs; makers of shoes and leather goods, clothing materials, machinery, and equipment; also railroad and steamship companies and other agencies of distribution; and, last but by no means least, bankers and financiers—all keep their hand on the pulse of agriculture, keep familiar with its needs, its methods, its accomplishments, its problems of production, fertilization, and pest control, and its periods of prosperity or depression. The structure of any one of the types of activity just listed is so closely interwoven with that of agriculture that even the expert finds it impossible to tell where the one leaves off and the other begins. The forces of commerce and industry in all lines of managerial and expert service are being drawn more and more from the ranks of those who are educated in agriculture, agricultural science, and agricultural engineering.

It is clear to all who grasp the significance of this picture that when a man seeks training in agriculture and related science, no one can tell in what ramification of industry or commerce he may land as a result of this training. It is equally evident that whatever mathematical training and knowledge are essential to the proper conduct of any line of activity allied to agriculture they are also, at least potentially and fundamentally, essential to training in agriculture and agricultural science.

Especially closely associated with the field of agricultural science are the various milling industries producing flour, foodstuffs, and clothing materials, as well as paper, lumber, and other building and structural materials. This combination creates an ever increasing demand for intimate knowledge of modern forest development, conservation, and management, of salesmanship, of mechanical knowledge and training, and most important of all, of financing, including such matters as investments, insurance, banking, property valuation, loans, and amortization. The range of mathematical knowledge required to handle this varied demand is readily seen to be all the way from the four fundamentals of common arithmetic up through the various ramifications of the compound interest principle, mensuration, logarithms, statistics, calculus, and the mechanics of materials.

The Rural Home Problem. Inseparable from the conduct of the farm business is the factor of the rural home which involves problems of housing and feeding adjustment as between the operator's family and the hired help, the simplification of labor through modern methods and devices, sanitation, the economics of food and clothing, and child training and welfare. Here the requirements in mathematical knowledge and training seem not to be so broad and varied, but they are none the less definite in the fields of proportion, mensuration, and minor financing.

Agricultural Science and Organized Research. This field comprises:

Agricultural Biochemistry, which to-day is at the very foundation of human food and clothing supply in the processing of dairy products, flour, and other grain foods, dye stuffs, and clothing materials; and also building materials developed from farm grown crops, and the like.

Soil Science, on which is based the conservation and development of soil fertility, and the conservation and control of soil moisture.

The Biological Sciences, which involve the development of desirable types and strains in plant and animal life, such, for example, as are resistant to diseases and climatic rigors—and the control of both plant and animal pests.

Agricultural Economics, which deals continually with production and price statistics, and market trends.

The educator and investigator in any of these lines of science to-day demands that students specializing in these fields and others assisting in them in research work shall be strongly fortified in mathematics, at least through the calculus and frequency distribution. For the worker in the field of genetics especially, such forms of elementary series as geometrical progression, fundamental in the study of life cycles, and the binomial theorem, the very foundation of the Mendelian law of inheritance, are essential parts of the mental equipment.

Agricultural Engineering. It is of course understood that engineering practice of any type is founded on mathematical and physical science. Agricultural engineering is in its very nature unquestionably the broadest branch of the engineering profession and embraces the following fields:

Farm Structures, covering farm buildings and other structures and related equipment, such as that for lighting, heating, ventilation, sanitary arrangements, and water supply.

Farm Mechanics, covering animal, steam, gas and electric power, tillage machines and implements, farm home conveniences, and related equipment.

Land Reclamation, covering land clearing, drainage, control of soil erosion, irrigation, and related structures and equipment.

The proper subdivision of the farm, or the planning of a farmhouse best adapted to the care and feeding of the help, or of a crop storage building, requires a knowledge of mensuration, surveying, and of the laws of statics fully commensurate with that required in planning a city addition, a city home, or a city office building, or factory.

The design of a plow bottom, a grain drill, a threshing machine, or a tractor requires quite as complete and thorough knowledge and understanding of strength of materials, laws of friction, and principles of mechanism, of thermodynamics and adiabatic expansion as does the design of a weaving loom, a sewing machine, a multiple cylinder printing press, or an automobile built for pleasure and speed, although these two great classes of problems are essentially different both in their character and in the ends sought; while the selection of the type of tractor best suited to do the work on a given type of farm under given local conditions requires a knowledge of structural materials, depreciation, soil dynamics, and labor and power distribution, which is unique in the field of engineering.

It is here interesting and significant to remember that Dr. E. A. White,[1] the first acknowledged agricultural engineer to receive a Doctor's degree in that field, developed some fifteen years ago as his Doctor's thesis "A Study of the Plow Bottom and Its Action upon the Furrow Slice,"[2] a scholarly treatise of 34 pages, 24 of which are pure mathematics of a high and difficult order. This study has had a profound influence on modern plow design and usage. It is the recognized forerunner of extensive studies, now being carried on at some of our agricultural experiment stations, on soil dynamics in relation to tillage as a guide to the efficient design of farm power units and tillage machines required in modern agriculture.

The intricacies of design called for in modern scientific planning of farm drainage and irrigation systems and of soil erosion control works, involving not only an intimate knowledge of the basic laws of water flow in closed conduits and open ditches but also of moisture relations of crops, of soil texture and soil water movement—soil hydraulics, if you please—call for a grasp of advanced phases of mathematics and mathematical physics in new and difficult paths that is even now taxing to the utmost the abilities of our leaders in mathematical and physical thought.

Education in Mathematics for Agriculture

This brief survey of the field of agriculture in its various ramifications brings us at once to grips with the question of what fundamental mathematics in secondary school education is needed to prepare for work in this field. In this connection reference may quite properly be made to a former paper on this subject read before the Mathematical Section of the Minnesota Educational Association in 1921.[3] The situation is not radically different from what it was at that time except that perhaps the demand for higher

[1] Director of the National Committee on The Relation of Electricity to Agriculture.

[2] *Journal of Agricultural Research*, No. 4, Vol. XII, pp. 149-182.

[3] "Minimum of Mathematical Requirement for Agricultural Study." *Mathematics Teacher*, Vol. XV, No. 1, January 1921.

mathematics in education for agriculture and related activities has grown much stronger.

Historical Retrospect. During the first decade of the present century this demand was at a very low ebb, the chief value of mathematical training for students in the agricultural field being considered by many leaders in that field as a necessary evil whose chief value was cultural and disciplinary. This attitude, however, resulted in such a decline in the general quality of scholarship in agriculture that by the middle of the second decade of the century a decided protest against neglect of mathematical requirement began to make itself felt and its more vigorous proponents started a definite study of, and a propaganda for its more utilitarian aspects. By 1918 this activity began to take definite form in the claim that what was really needed was a general mathematics especially adapted to the elementary requirements of agriculture. All non-essentials were to be swept away and only those topics were to be retained which could be shown to have a direct bearing upon some phase of agricultural work, supported, of course, by the recognized primary fundamentals.

Within three or four years several new texts appeared purporting to meet the requirements just stated. The general plan seemed to be clothed with official sanction when the Resident Teaching Section of the Association of Land Grant Colleges and Universities, in its annual convention of 1920, voted approval of a syllabus for a suitable text for such courses, which was presented to the session, in galley proof form, by Professor Samuel E. Rasor of the Department of Mathematics of Ohio State University.

A Specialized Mathematics for Agriculture. There seems to have been quite general agreement as to the requisite details of such a course, these being in general about as follows:

A review of the basic essentials of elementary algebra as given in standard secondary schools through quadratic equations.

The elementary principles and practice of common logarithms.

Elementary series, including especially the progressions and the binomial formula.

Depreciation and elementary accounting.

The compound interest principle and annuities, perhaps better thought of as the mathematics of investment.

Ratio, proportion and variation, dairy arithmetic, and mixtures.

The principles and practice of numerical trigonometry, especially as applied to general mensuration and plane surveying.

Permutations and combinations.

Elementary statistics, frequency distribution, and the elementary theory of errors.

Consideration has also been given to various combinations of certain of these details, under headings of certain fields in agriculture, such as, for example, the mathematics of farm management. The inclusion of this type of material has received considerable support from the specialists in the farm management field.

There has been a tendency on the part of some of the earlier text writers to include topics that belong specifically to other fields, such as drawing, physics, and surveying. This practice is not well advised, as each of these is a field by itself that holds and requires fully as prominent and important a place in our school and college curricula as does mathematics. While these fields are among the richest sources of real problems in mathematics, the inclusion in a mathematics text of chapters on pure physics, drawing, or surveying is to be deprecated, as no one of these distinct fields of study or even of their major subdivisions can be adequately covered in a chapter or two. This seems especially true when it is remembered that the time allotted to mathematics in school practice is frequently too limited for thoroughgoing results even in that field alone.

Difficulties Encountered. The idea of a specialized mathematics of agriculture has grown up with such startling suddenness that, as might be expected, mistakes in the plan of operation have seriously interfered with its complete success in practice. Two of these mistakes stand out with unusual clearness.

1. College administrators, continually harassed by the demands of numerous new types of courses for admission to the required curriculum, and often influenced by those who cherish a lifelong and unreasoning antipathy toward mathematics, have seized upon the new idea of mathematics in agriculture as an excuse to limit the time allotted to mathematics in agricultural curricula to a degree utterly inadequate to the ground that must be covered. In many cases this has been carried to the extreme of limiting the time allot. .d to mathematics in the agricultural college to one quarter, with three to five hours per week, when those responsible

for the conduct of the work are painfully aware that three to five class hours a week for a full year is little enough time in which to expect the general run of freshman or even sophomore students to grasp a working knowledge of the mass of topics called for in such a course. No doubt the administration justifies its attitude by the plea that the useless lumber of old-fashioned mathematics, comprising fully two-thirds of the subject matter material, has been cut away, and that i. should be possible to cut the allotted time correspondingly; a not well digested argument to be sure, but intrenched behind a strong rampart of power.

2. Too often the teacher assigned to the course is selected because of his training in the field of pure mathematics, and from the general faculty of the department of mathematics. In such cases the chances are that, although he may be well trained in mathematics and even though he may be an excellent instructor, he knows little of the field of agriculture and his vision o. the relation of mathematics to the agricultural field and its many-sided functions may be represented by the minus sign. Regardless of his scholarly attainments, his pedagogical talent, or his enthusiasm for pure mathematics as such, a teacher of this type will find it very difficult to elicit the interest and support of the various departments and groups of an agricultural faculty so essential to success in teaching the mathematics of agriculture, and he is thus very apt to be deprived of his most reliable source of real problems which show the relationship of the pure fundamental science to the applied.

Selection of Teachers. It is a well-established practice in engineering schools and colleges, based on experience, to select as teachers of mathematics men who have been trained as engineers, and are, as far as possible, experienced engineers with a natural grasp of the relations of pure mathematics to the problems of engineering. In the light of that experience and with the fuller vision of the place of mathematics in agriculture that is opening up to the present age, why is it not the sane and logical procedure to select our teachers of mathematics for agriculture more largely from the ranks of those agriculturists, agricultural scientists, and agricultural engineers who have a clear understanding both of pure mathematics as such and of its relation to the problems of agriculture? This would seem to be a rational practice, not only for our colleges but also for those secondary schools where preparation for advanced study in agriculture and in allied lines is a pronounced feature.

In fact, in view of the time limitations on mathematics in the junior agricultural college curriculum, it would seem that the ultimate success of any type of instruction or any type of course in mathematics of agriculture must rest largely with the teacher of mathematics in the secondary school.

Subject Matter Requirements. This thought then brings even more definitely before us the fundamental question: What are the secondary school requirements in mathematical study that serve as a suitable preparation for advanced education in the agricultural field? An additional decade of experience in this field has not greatly changed the answer from the one given ten years ago.[4] The extent of mathematical training needed is largely dependent upon the particular student's line of specialization in the field of agricultural or related study, but for good results the secondary school mathematics should, without doubt, be much the same in all cases. It does not appear that its outer boundaries should greatly differ from those usually existing, in theory, in our established grade and high school curricula. However, possibilities of improvement over the stereotyped old order in interior subdivision, topic arrangement, content, and methods of presentation unquestionably exist.

Arithmetic. The young people of the present generation are likely to be bunglers in their work with common arithmetic. Too much stress, therefore, cannot be laid on extensive practice in the earlier grades in the use of the four fundamental operations; and the numerous short cuts in multiplication and division should be known and used with facility by every normal young American. It is wholly unnecessary to find a supposedly normal child of high school age and standing using the process of long division in dividing by any single digit from 2 to 9, and no teacher should accept or lightly pass over such a piece of work.

Teachers in other fields of work than mathematics, in which mathematics is a necessary tool, complain that the students are incompetent to do work involving the common operations in fractions, decimals, percentage, and simple mensuration. The responsibility for this situation cannot be laid upon the children themselves; it rests upon the teacher.

[4] *Mathematics Teacher*, Vol. XV, No. 1, January 1921. *Loc. cit.*

Considerable stress is laid by leaders to-day on intuitive or informal geometry as though it were a new idea in mathematical training and usage. It is not new except perhaps in name. The arithmetic texts of forty or fifty years ago included a very considerable chapter on mensuration. There are still living thousands of our older people who received the fundamentals of their education in the one-room, ungraded, country school of an earlier day, and who found both intellectual profit and keen delight in the mastery and practice of both planimetric and volumetric mensuration, long before they ever heard of geometry as such. Many of them who never heard of formal geometry, as now taught in our high schools, can apply their early training in mensuration to the everyday problems of capacity, area, and direction with greater facility than can the majority of their children and grandchildren who passed through exposure to arithmetic without any conscious contact with mensuration and who consider the mastery of geometrical principles to be the epitome of drudgery and dullness instead of the acquisition of new mental power which it is.

In brief, then, the teacher of arithmetic should insist on a ready and correct use of the fundamentals in practical applications. Intuitive or informal geometry is as properly a part of arithmetical training as is percentage or interest, and beyond that it is a fruitful source of practical problems to which no student in the agricultural field should be a stranger.

Algebra. As elementary algebra is only generalized arithmetic the beginnings of algebraic notation and processes, such as the term, the exponent, the simple algebraic fraction, the binomial, and even the simpler special products and the simple equation in one unknown, should be introduced to the student of arithmetic at the earliest possible moment. There is no reason whatever why this practice should not begin at least as early as in the seventh grade and possibly in the sixth. Then by the time the pupil reaches formal algebra in the eighth or ninth grade he will be ready for it and the transition from arithmetical methods of thinking in number will be gradual and natural and carry with it no depressing mental shock. This is as it should be and is in line with progressive thought in education. Dr. Reeve's quotation from Dr. Charles W. Eliot [5] sums up and clinches this whole matter in excellent fashion.

[5] *The Fourth Yearbook, National Council of Teachers of Mathematics*, p. 141. Bureau of Publications, Teachers College, Columbia University.

In regard to requirements in elementary algebra, this point has been so definitely covered in the former paper already twice referred to that it needs not to be discussed here, although some additional comment may be justified. It now seems clear that the more complicated factorable form $a^n \pm b^n$ should be among those omitted from high school work in algebra for it finds very few applications in practical applied mathematics. The time that would be spent on it by the high school pupil can be spent much more profitably on other parts of the subject.

One bit of repetition from the earlier discussion seems amply justified, even though it is in direct opposition to eminent modern authority. Treating a proportion as an equation of simple fractions is, of course, perfectly proper, for that is what it is, but it must also be treated and taught *as a proportion*. The principles of proportion are essential, simple, and eminently useful. The proportion, as such, is much more readily applied than is the algebraic equation of simple fractions. Proportion is the front door to variation and both are essential in chemistry, in physics, and in engineering. The teacher of mathematics will not secure and retain the whole-hearted coöperation of the scientist and the engineer by refusing to teach the principles of elementary mathematics for which these groups have daily and constant use. The chemistry of to-day is hard for the average beginner anyway. Whether the administration is willing to admit it or not, beginning chemistry is an elimination course in the freshman year in many of our colleges. The teacher will make it doubly hard for the average freshman and almost insure his elimination, or, at least, his strict probation, by refusing to equip him with a comprehensive and thorough knowledge of proportion and variation; and this refusal will not magnify the teacher in the esteem of the student.

Elementary series, particularly the progressions and the binomial theorem, are the very foundations of much biological science as related to agriculture. Therefore, if they are not taught, and thoroughly, in the secondary school they will have to be taught in the junior college. Is such a plan good pedagogy? Is it not rather a waste of valuable time for the student to have to secure, in the junior college, essential mental equipment that he should already have when he first enters there?

One feels an urge to enlarge on the place of the graph in algebra, especially when he considers its multiform uses in agriculture, but, as far as the subject is related to the teaching of mathematics, Dr Reeve effectively sums up the essential idea as follows:

The study of the graph is a major trend to-day in algebra because, with the formula, it helps to clarify the idea of functionality. We now emphasize the meaning of graphs rather than the making of them. . . . Owing to the prominence of the statistical graph, and the increased interest in educational statistics, graphic work is assured a permanent place in our courses in mathematics.[6]

It might be added, "particularly in those on the mathematics of agriculture."

As algebra is basic to all other mathematics it may be advisable to emphasize the fact that the high school student should be so well trained in it that he handles its elementary principles with understanding and facility. If the customary year allotted to elementary algebra in high school curricula is not enough to produce such a result, an extra half year of *advanced* elementary algebra, treating of the more difficult forms and including additional topics such as the progressions, variations, and logarithms is desirable and preferable to so-called high school higher algebra with its abstract proofs mostly beyond the intellectual grasp of the secondary school student. For the young student facility in application comes from handling often, with familiarity, rather than from deep reasoning. Algebraic demonstration largely calls for greater mental maturity than is usually found in the high school student. The practice of some of the best high schools in offering the suggested extra half year of advanced elementary algebra or general mathematics based on algebra, in lieu of the half year of so-called higher algebra, has been productive of good results and is to be commended.

Geometry. Elementary fundamental requirements in geometry, as in arithmetic, algebra, and possibly trigonometry, are much the same whether the ultimate field of the student's life work be that of medicine, law, engineering, theology, or agriculture; for as one graduate school dean was recently heard to state, advanced research in applied science can be naturally and safely based only on work in the fundamental sciences of mathematics, physics, and chemistry, with particular reference to their application in solving the given special problem in the given field of applied science.

The fundamental requirements in geometry in secondary school

[6] *Fourth Yearbook, National Council of Teachers of Mathematics* p. 160.

training have been already so carefully covered by Dr. Reeve, by the writer, and by others, that extended additional discussion here is unnecessary. It may be well, however, to place the emphasis of repetition on the following basic considerations.

The number of fundamental propositions demonstrated in the text should be reduced to the minimum absolutely necessary as a framework for the subject.

Original demonstrations should be encouraged and simplicity and conciseness therein should be insisted on.

No sharp line should be drawn between plane and solid geometry. They can and should be taught in a combined course that need not exceed one school year in length.

It has never been well demonstrated that trigonometry belongs in the high school curriculum, particularly in the college preparatory courses. If taught there at all it should be limited to those simpler phases directly applicable to the solution of practical problems and its presentation should be made a part of the work in plane geometry. It is not that elementary numerical trigonometry is too difficult to be grasped by the mind of the high school student but rather that, as a rule, when taken as a complete subject in the high school, plane trigonometry is seldom covered in its entirety. The student having been once exposed to it in his high school experience generally passes by the college course in trigonometry with its broader applications and point of view with the result that he is usually lame throughout his study of analytic geometry and the calculus, even though by the time he has completed these subjects he may have attained a mastery that enables him to handle still more advanced phases of mathematics with credit.

The value of practice in application of geometry to the solution of practical problems cannot be overemphasized, both as a stimulus to the interest of the student and as a valuable lesson in the practical utility of fundamental science in everyday affairs.

Calculus in High School Courses. Right here it is of interest to note the discussion by Dr. Reeve of "Calculus in the High School," together with his inclusion of elementary calculus in the outline of a prospective course in mathematics for twelfth grade (senior high school) students.[7] This is perhaps a bold step, but

[7] *Fourth Yearbook, National Council of Teachers of Mathematics*, pp. 164 and 173

within the limits of available time it must be conceded to be allowable for the idea and the use of the first derivative, for example, are not difficult to grasp, and experience has shown that it frequently clarifies ideas commonly included in the elementary mathematics of high school curricula which the students find difficult to handle or to understand fully without the aid of the derivative. The subject of maxima and minima is a clear case in point.

The writer cannot agree with the idea that daily work by the class at the blackboard is a mistake. Practice of the blackboard type with a well-organized class, when consistently followed in a well-planned course in geometry and supplemented by rigid questioning and criticism by both class and teacher, uncovers and breaks down the bad habit of memorizing proofs, stimulates keenness of insight, power to visualize, and clarity of logic. The best results in the teaching of geometry have followed systematic daily board work, broken occasionally by special presentation of originals or particularly complicated regular theorems by selected or by volunteer members of the class. If any great amount of time is wasted in connection with board work it is unquestionably due to lack of system and class organization for which the teacher alone is responsible.

VALUE OF UNITY AND THE UTILITARIAN POINT OF VIEW

In closing this discussion it seems desirable, even at the risk of some repetition, to outline a certain broad foundation on which to base the requirements in mathematical study for any field of applied science. This, of course, includes modern agriculture, whether considered directly or through the avenues of its many closely related fields of interest.

Mathematics Inseparable from Human Experience. It has always been difficult to understand the antagonism displayed by many people to the idea of acquiring mathematical knowledge and understanding; for, whether or not one recognizes or acknowledges the fact, we live in a scientific cosmos in which the fundamental, governing forces all act in accordance with definite law. Law is the idea, whether expressed or only implied, of the method of orderly progress, and orderly progress is mathematical in its very essence. Hence, no one can escape the continual influence of mathematical principle in his life experience, no matter how much he may claim or seek to do so.

Unity of the Science the Key to Success. However, mathematicians and other scientists should recognise that, among a large mass of people, they have to meet this antagonistic state of mind. It seems reasonable to assume that the simpler, the more unified, and the more directly applicable to the problems and interests of daily life the science of mathematics can be made to appear to the public mind, the more successful will they be in overcoming this antagonism.

We must, therefore, think of mathematics, not as arithmetic, trigonometry, or calculus, not as algebra, frequency distribution, or theoretical mechanics, but as the science of quantity and its relationships to thought and action with all their material accompaniments. In other words, in our general consideration of the science of mathematics in relation to human life, education, and progress, we must wipe out the rigid artificial boundaries set up through centuries of stilted and pedantic scholasticism between what for want of a better term we are wont to call "the branches of mathematics," for no such lines of cleavage exist in scientific fact.

Utility Demanded by the Youthful Mind. One of the essentials in mathematical training that applies with especial force to the broad field of agriculture is the development of its eminently utilitarian aspects. That is, we must seek continually to develop the ability of the student to apply mathematical principles to the phase of human interest.

To accomplish this end in these times of swift progress, wherein both physical and intellectual inertia are being more fully overcome almost daily by new and startling developments that have nearly eliminated time and space in human experience, we must be ready to enter into the spirit of present-day youth. This means that we must always be on the alert to secure *real* problems from any and every part of the field of activity. We must present them to our students not alone for practice in their solution—which must be insisted upon—but even more to arouse their interest and to show the utilitarian side of mathematics in practically every phase of human interest.

Old Type Problems Obsolete. The youth of to-day from kindergarten to college commencement is little interested in the infantile, senseless, or purely artificial problems with which our mathematics texts from arithmetic to calculus have been wont to be packed. Rather is he interested in what he sees actually going on around him everywhere, things and events that result in activity, in growth evident to the eye, in enjoyment, and whatever leads thereto. The youth of to-day is far more sophisticated than was either his father or his grandfather at the same age, and the teacher of mathematics must play up to this fact.

Meeting the Spirit of Youth. A boy is not interested to-day in how many oranges at 5 cents apiece and how many marbles at 3 cents apiece Mary can buy at the corner store for 25 cents, nor yet in the number of leaps the hound of fable must take to catch the equally legendary hare. His interest lies in the things he reads in the headlines of the daily paper and the things that he hears older men discuss—the baseball score of the major leagues, the stock market, the price and speed of the Ford coupe, or the range of the radio. This state of mind is especially characteristic of the youth in contact with things agricultural. He, too, follows the baseball score and the automobile, but he is interested also in the price of wheat, the tractor *versus* the horse as the power that draws the plow, the binder and the combine, the influence of the tariff on the price of pork and how it affects his profits on his pet litter of spring pigs, and the latest developments in the airplane, which has recently become an agricultural machine engaged in the warfare against plant and animal pests and diseases. Yet where would the tractor, the radio, or the airplane be to-day but for the mathematical principles involved in their design, construction, and operation—harnessed and put to work by the mind of man?

Just so far as the teacher of mathematics looks for his problem material to fields of human interest, especially those of particular interest to the youthful mind, is he likely to succeed in the task that lies before him to-day in fitting his science to the needs of all industries, including agriculture.

A Few Well Chosen Problems Best. One last thought and this argument is closed. In any topic in the field of mathematics, a *few* problems, carefully chosen along the lines herein indicated and illustrating definite points at issue in a practical and interesting way, especially if these problems be fully analyzed and clarified in the mind of the student, are far more likely to fix his attention and stimulate him to mastery of the principles involved than is a great mass of problems less alive with human interest. Multiplicity of problems under any new topic serves only to confuse the student mind, much theoretical authority to the contrary notwithstanding.

MATHEMATICS IN PHARMACY AND IN ALLIED PROFESSIONS *

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Introduction. The Yearbook is doubtless the proper place for a presentation of the usefulness of mathematics in pharmacy, medicine, dentistry, and nursing, inasmuch as it will be read by those most interested in the preparation of young men and women who are to enter these professions.

This chapter will not deal with mathematics needed by the research worker in the professions mentioned, but will confine itself to the mathematics useful to the study and practices of the professions.

The Mathematics Curriculum in the Professional Schools. It may be of interest in the beginning to state that little or no mathematics, as such, is taught to the medical student after he enters the medical college, for it is presumed that he is properly prepared to carry out all necessary calculations confronting him in his work. The dental student in some instances is taught some application of mathematics in his course in materia medica. The nursing student is taught some mathematics in the course known as "Drugs and Solutions." Nearly every pharmacy school to-day offers a course in pharmaceutical mathematics. The teaching of applied mathematics in these curricula which has come about during recent years is due, I believe, rather to the growing consciousness of the importance of the subject than merely to the unpreparedness of the entering students.

It would be an easy matter to shift the responsibility for the lack of knowledge of simple arithmetic and algebra to the high school and grade school teachers; but is it not more reasonable to assume that the student entering a profession will probably be more

* Readers who are interested in finding further problems of the kind herein discussed should consult *Pharmaceutical Mathematics* by the author of this article and published by McGraw-Hill.—THE EDITOR.

or less unskilled in the mathematics of his childhood and its application in the professions? Should we not rather expect him to be taught the application when he reaches the study of the profession, and to have not only ability but also alertness enough to review what he most needs?

Nearly every student entering a medical college to-day will have had at least one year of college mathematics, and many of them will have had some calculus. This will be true also of the student of pharmacy before he has completed his college course, as most of the colleges of pharmacy give at least one year of mathematics in the four-year course. This year will embrace college algebra, trigonometry, and analytical geometry. Some colleges restrict this year to the first two subjects mentioned and some to the last two.

The four-year course will be the minimum one for Association Schools of Pharmacy after 1932. Such mathematics, especially in university schools or where a connection with a college of liberal arts is possible, is taught by a teacher of mathematics or is given in the regularly prescribed courses with students of liberal arts, sciences, or engineering.

In addition to the above mathematics, all students of chemistry must know and use the applied mathematics of chemistry. The mathematics of the freshman year of chemistry is comparatively simple, but the student of the sophomore or junior year will take a course in chemical problems, either as a separate course or interwoven into the course in quantitative chemistry. Most of the students of medicine and dentistry will have this work before entering upon purely professional studies, though quantitative chemistry is not in all instances required of them; and all students of pharmacy will have it early in their professional work. The difficult features of such a course are seldom found in the pure mathematics, but more often in the application to the field of chemistry.

It appears to-day that the weakness of the students coming to us is in simple arithmetic and in very simple algebra, but it is also true that it is easier to teach the applied mathematics of chemistry and pharmacy to students to-day than it was fifteen years ago. The reader may draw his own inferences.

To summarize what has gone before and up to this point one may say that the student of medicine, dentistry, and pharmacy will have had the mathematics of grade school and high school and at least college algebra, trigonometry, and analytical geometry, and some will have had both differential and integral calculus. The student of nursing entering upon the combined courses now offered in some universities will in some instances take college algebra and trigonometry.

Prerequisites in Mathematics. Class A medical schools require two years of liberal arts work for entrance, and although mathematics is not required in most schools, it is suggested as being a useful background course. Some medical schools require three years of liberal arts work and others four. The four-year course in pharmacy in nearly all instances embraces the study of at least one year of college mathematics exclusive of the applied mathematics of pharmacy and chemistry.

Any student who is not properly prepared in mathematics will experience difficulty in any of these four professions. The best advice that a high school teacher can give to a boy or girl planning to enter any one of these professions is to take all the mathematics his high school offers. I do not say definitely that a student who has not had all the mathematics available in a high school will fail in one of these professions. I do say that I have never known one who has had such work to fail and I do know that the student who is good in mathematics and who has taken much of it finds his professional work far easier.

Applied Mathematics of Pharmacy and the Allied Professions. The applied mathematics of the four professions is very similar, though perhaps the pharmacist finds a range greater than that of the others; so the subject from this point on will be discussed from the basis of the elementary mathematics employed and its application. No attempt will be made to classify it for each or any one of these professions. The above statement is of course made without reference to the mathematics necessary for the research worker.

First are taught tables of weight and measure in common use and how to change denominations from one table into like denominations in another. A student of any of these professions or a student of chemistry who cannot understand and make these transpositions easily and quickly is handicapped. The metric system is now accented but the student must be familiar with the avoirdupois, troy, and apothecaries' weights as well. In pharmacy, laboratory work accompanies or follows this study and practically all calculations found in the mathematics course are carried out in the laboratory.

The writer has seen the following experiment made with students of medicine and of dentistry and the interest displayed by them was positively amazing. Weigh a grain of wheat, using apothecaries' weights, and then weigh it upon a fine analytical balance, using metric weights. The fact that one grain is equivalent to nearly 65 milligrams has taught some concept both of the grain and its origin and of the size of the milligram. I shall leave it to the reader to decide where this instruction belongs, whether in grade school or in college. I believe I can argue upon either side of the question but all will surely agree that the student of these professions needs the knowledge. It may be an innocent sport, if one be interested in this subject, to ask his physician, his dentist, his pharmacist, or his nurse as he meets them, What is a grain? What is a milligram? The writer once had a class of fifty freshmen, twenty-two of whom had never seen a grain of wheat.

In addition to an understanding of the tables of weights and measures and the transposition referred to, it becomes necessary for the student to develop practical knowledge so that answers to given problems are weighable and measurable with the usual apparatus at hand. This is a practice that may well start in the early study of arithmetic. A result involving a weight, volume, or linear measure should always be stated in such terms or denominations that it is of practical usefulness.

As an example of an improper answer for a problem I present one which I once received from a state department. I had asked for a definite location of a culvert on a canal. The map showed it to be a short distance from a bridge and also a short distance from a canal, called a "side cut," leaving the main body of water. The answer given me was somewhat like this: "446,700 feet from the north corporation line of ———" (a town several counties away). The point I wished to locate was just a trifle more than a mile from the "side cut."

The review of ratio and proportion is always necessary, as it is a short cut to the solution of many everyday problems. The term "ratio" and the mathematical expression of it not only are found in our daily laboratory work but will be found in a vast number of the texts and articles read by the four professions in everyday life.

I am not familiar with the method employed in preparatory schools to teach this subject, and I sympathize with the teacher who must teach it; for I find there are some students who can never really grasp it, simple though it is. Students who come to us from city schools seem fairly familiar with the fractional method of expressing a ratio, but have not been taught the real meaning of a proportion or that it may be stated in whole numbers as well as in fractional form. Ratio and proportion are used in both chemical and pharmaceutical calculations and are well-nigh indispensable.

The reading and writing of fractions and the transposition of the common fraction to the decimal and vice versa are almost daily occurrences.

To the student who cannot master relative sizes, volumes, weights, and other dimensions and who does not have a mental picture of fundamental units, such things as the capacity of bottles or laboratory glassware constitute a never-ending problem.

An examination was given to a number of recent high school graduates. It may be of interest to observe that they came from large city, small city, and small town high schools. They were given a problem which is not an uncommon one in everyday practice—to calculate the volume in gallons of a drum (cylinder). The figure of 231 cubic inches in a gallon was furnished, but not the rule. They all knew the rule and how to apply it. The dimensions of the drum were stated to be 36 inches in height and 18 inches in diameter. One student's answer was "Four gallons." He had made a mistake in calculation but did not observe that almost anyone could hazard a guess or make an estimate that would be closer to the real answer.

Some practice, in addition to that of actually proving results, should be given to the end that a student will not present a result that is ridiculous. It is needless to offer more than the above suggestion to an intelligent teacher as to the field open in the laboratory for the use of mathematics.

The subject of percentage is very important and it likewise is a much abused subject in these professions. We use not only true percentage but approximate percentages and near percentages; and therefore, unless its real meaning be perfectly clear to the student, much confusion results. It is very doubtful if the student exists who cannot calculate 5% or 6% of any given number of dollars or any given number of pounds, but the converse of even such a simple problem always seems difficult. The application of percentage to business problems, to percentage strength of compounds and to the mixing of substances of different strengths, to the dilution of substances to a definite percentage strength, and to the fortification of others always seems to mean something different from the old friend, percentage, of grade school days. There are, of course, some things that interfere with the mixing of substances of different percentage strengths that complicate what is otherwise a simple mathematics problem. For example, one hundred parts by volume of 95% by volume alcohol mixed with 100 parts by volume of water will only make about 190 parts by volume instead of 200 parts. Such procedures certainly should be taught in an applied course and not in a preliminary course confined to pure mathematics. It would be helpful, however, if teachers could employ percentage in other ways than the relationship of it to financial considerations. Why not study ratio and proportion in mixtures such as concrete, one cement and three sand, or cement, sand, and crushed stone, and then express them in terms of percentage? How about volatile matter and ash from coal? Such examples would help us in percentage purity of chemicals, percentage composition of compounds, and percentage strength of solutions.

Under the heading of the calculation of dosage we make use of both common and decimal fractions and it is often necessary to add, subtract, multiply, and divide them. The average student has usually forgotten how, but it does not take long to bring back this knowledge, and my observation is that once back, it stays, because it is in daily use.

A nurse might be confronted with the problem of giving a $^1/_{10}$ grain dose of a substance and all that she has are tablets of the substance, each containing $\frac{1}{4}$ of a grain. Fractions first, of course, and then knowledge that a tablet cannot be divided with safety, that small amounts cannot be weighed accurately upon the apparatus at her disposal, and so on *ad infinitum*.

If time is not too short the student should be taught why $\frac{1}{3} + \frac{1}{4} = {}^7/_{12}$ and why the common denominator is used. I here mean to express that the student knows how to make such calculations if he has not forgotten mechanical instructions, but he does not seem able to see practical everyday problems and usage.

He views these calculations as school tasks of to-day and does not see their relationship to life problems.

When it comes to calculating specific gravities of solids and liquids, the laboratory helps the student's mathematics. The old-fashioned drill of giving two factors in a three-factor product to find the third often simplifies the entire teaching of this subject. After the student learns what specific gravity is and begins to use it, his mathematics becomes clear. This is a subject which he has usually had in high school physics or in grade school mathematics, and sometimes even in college physics, but its value has always been academic to him.

Calculation of allowable errors in weighing and measuring is always more or less difficult. This again may often be complicated by percentage. Suppose a student is told to prepare five gallons of a mixture so that one teaspoonful will contain $^1/_{60}$ of a grain of strychnine sulphate. How accurate must be the balance upon which the strychnine is weighed, and what percentage of error in either direction does his good judgment tell him is allowable? Here again is applied mathematics. Percentage solutions, solutions made on a ratio basis, and saturated solutions all involve something besides mathematics, but the simple mathematics must be known first. The difficulties usually involved in teaching conversion of temperature scale readings are obviated by teaching these readings as definite measurements of length. Interpolation in specific gravity tables becomes easy to the student who has used logarithms and this last subject is one that is needed by the student of medicine and pharmacy very often indeed. Old-time alligation has to be taught pharmacy students because State Boards demand it. The writer, however, is not an advocate of its use.

A student always needs to understand problems of interest, discount, profit and loss, and many other problems found in the average commercial arithmetic.

The Preparation Students Should Have. The writer feels that he will not do justice to himself unless in closing he states his own opinion upon the preparation of students in mathematics in order to make professional work easier. That opinion is that arithmetic should continue into the high schools, either in the form of the old-fashioned advanced arithmetic or in the form of review in the senior year. I make this statement knowing full well the objections I should hear if I happened to be a well-known teacher of mathematics. I hasten to add, however, that arithmetic should not replace the algebra, geometry, and in some instances higher mathematics now taught; it should be taught in addition to them. A student expecting to study any one of the four professions which we have discussed, or one expecting to study chemistry or physics, needs mathematics every day.

Much of the necessary arithmetic may be reviewed by the teacher of algebra, and may I add here that the teacher of algebra should endeavor at all times to show, where possible, that much of algebra is really a review of arithmetic and is after all an easier method for solving many simple problems. The simple equation, to find the value of x, still seems to be difficult for many students to solve. They still find it difficult to form a simple equation and solve it, and for such students the use of the simultaneous equation is out of the question.

In my own institution we were once confronted with the problem that our required course, embracing mathematics, was too severe; that students who did not wish to become scientists and real professional men should be permitted to take economics instead of mathematics and language. It was their desire to know merely enough of science to be safe technicians, and otherwise they would have purely business interests. It soon developed that those who could not pass in mathematics felt they could join this second class, and so our poor students went to the economics classes and did not endear us to the professor of economics. Before long we discontinued this practice, for we are of the opinion that business as well as science needs men of brains who have a working knowledge of mathematics.

I must plead guilty to belonging to that class of old-fashioned people who, if given the chance, would require for high school graduation arithmetic, algebra (three semesters), plane and solid geometry, and as much more as could be put into the course. Some of the writers upon mathematics have felt that a knowledge of mathematics and an ability to think do not necessarily go hand in hand and that the first does not tend to develop the latter; but those of us at this end of the scale are usually fairly certain that when we get a student who is a good student of mathematics and has been well grounded in the subject, most of our troubles fade into nothingness so far as he is concerned. May I add that I do not approve of applied subjects of any kind until after the fundamentals have been taught? There is always the danger of slighting the fundamental for the more interesting application.

I wish the specialist, the real mathematics teacher, to give the student his elementary training. I know the specialist will do this part of the work well, and I can then find the means to teach the student the practical and professional application of the mathematics he has learned.

MATHEMATICS AND STATISTICS

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I. Statistical Method in Present-Day Thinking

Importance of Statistical Method. More and more the modern temper relies upon statistical method in its attempts to understand and to chart the workings of the world in which we live. Particularly in those sciences which deal with human beings, whether in their physical and biological aspects or in their social, economic, and psychological relations, the spirit of our time asks that its conclusions be based not so much upon the distinctive reactions of one or two individuals as upon the observation of large numbers of individuals, the measurement of their common likenesses and the extent of their diversity. As the data thus gathered from mass phenomena become extensive, it becomes imperative to have methods of organization to bring the facts within the compass of our understanding, methods of analysis to make the essential relations appear out of the mass of detail in which they are hidden, and methods of classification and description to facilitate the presentation of the data for the study and consideration of other persons. Thus statistical method becomes a telescope through which we can study a larger terrain than would be accessible to our unaided vision.

Use of Numerical Data. As the area of investigation is widened to include larger masses of individuals and as the nature of the inquiry becomes more precise, it is inevitable that data and conclusions shall assume numerical form. To quote Sir Francis Galton:

> General impressions are never to be trusted. Unfortunately when they are of long standing they become fixed rules of life, and assume a prescriptive right not to be questioned. Consequently those who are not accustomed to original inquiry entertain a hatred and a horror of statistics. They cannot endure the idea of submitting their sacred impressions to cold-blooded verification. But it is the triumph of scientific men to rise superior to such superstitions, to devise tests by which the value of beliefs may be ascertained,

and to feel sufficiently masters of themselves to discard contemptuously whatever may be found untrue.

Sir Arthur Newsholme writes:

As the scope of a science widens, it is generally found necessary sooner or later to adopt numerical standards of comparison. In medical science this is found to be especially necessary, though perhaps in no other science is the difficulty of exact numerical statement so great. The value of *experience*, founded on an accumulation of individual facts, varies greatly according to the character of the observer. As Dr. Guy has put it: "The *sometimes* of the cautious is the *often* of the sanguine, the *always* of the empiric, and the *never* of the sceptic; while the numbers 1, 10, 100, and 1,000 have the same meaning for all mankind."

Adolphe Quetelet, the great Belgian astronomer, mathematician, anthropometrist, economist, and statistician, in the first lecture of a course on the history of science, said:

The more advanced the sciences have become, the more they have tended to enter the domain of mathematics, which is a sort of center towards which they converge. We can judge of the perfection to which a science has come by the facility, more or less great, by which it can be approached by calculation.

Relation of Statistical Method and Statistical Theory. Clearly, then, statistical method must be grounded in statistical theory, which is essentially a branch of mathematics. Indeed, statistical theory has its roots in the mathematical theory of probability and the work of the mathematical astronomers, notably Gauss and Laplace, who early in the nineteenth century built up a theory of errors of observation in the physical sciences. Statistical method is completely dependent on statistical theory, yet the two have important differences in purpose, in procedure, in technique, and in the type of talent and preparation needed for successful prosecution.

Statistical theory is developed for an ideal situation seldom completely realized in practice. Statistical method almost always involves a measure of compromise between the recalcitrant facts which life presents for analysis and a mathematical theory which postulates a particular form of distribution or other ideal circumstances only approximated by the data. The dependable statistician recognizes that the assumptions implicit in his formulas are not completely fulfilled, but he sees that to use these formulas and continue the investigation will afford a far closer approach to the truth he seeks than mere conjecture and intuitio:., which may be the only alternative.

Difference Between Statistical Method and Mathematics. The man who develops a new piece of statistical theory works as a mathematician and faces only those obligations ordinarily incumbent upon the mathematician. He must state his assumptions, he must avoid contradictory assumptions, he must be careful that his conclusions follow logically from his premises. Beyond that point, he is free to make whatever assumptions are convenient for the simplification of his argument. The worker in statistical method who applies to the solution of some practical problem a formula thus developed by the mathematician has an additional obligation. He must not only know the assumptions on which the formula rests, but he must also know the content of the field in which he is working well enough to determine whether these assumptions can be appropriately made in that particular situation. It is probably not essential that he be able to go through the steps of the derivation of each formula, but unless he knows what assumptions were made in that derivation and unless he ascertains that these assumptions can be reasonably made for his data, there is a possibility that he may come out with conclusions which are far from correct. Herein lies the weakness of many statistical investigations, either that the research worker does not know the mathematical theory well enough to recognize the assumptions upon which his procedure rests, or that he is not sufficiently at home in the field of research to pass upon the validity of those hypotheses.

Knowledge of statistical theory is not enough for the man who would plan important statistical investigations. Neither the pure mathematician nor the man innocent of mathematical training makes the best worker in practical statistics. The expert in statistical theory needs also a rich knowledge of the content of the field in which he would work. The general method of statistics is the same for all fields and the elementary training need not differ much whether a man is to work in biology or psychology, in economics or education. Therefore it is sometimes suggested that a consulting statistician trained primarily in pure mathematics and in statistical theory may act as consultant for a large number of important statistical studies in various fields, the data for which are collected by others and the results worked out by others. By providing expert advice on the method of research, such a man might make it possible for important studies to be carried out by men interested in the content but ignorant of statistical method.

This solution has serious drawbacks. Without a general knowledge of the field of study, it is difficult to choose appropriate statistical procedure. Without a consuming interest in the outcome of the particular study and an intimate knowledge of its details, fruitful leads do not arise, "hunches" are lacking, and the most significant facts and relations may be overlooked. Recently a research worker in biology wrote to ask me if he was justified in using a certain procedure. Not being a biologist I could offer no creative suggestions, and could only say, "The assumptions underlying the formula you mention are thus and so. I should be suspicious of them, but a biologist will have to pass upon their applicability."

On the other hand, the view is common that the mathematician who develops a formula has welded a tool which the nonmathematical psychologist or economist or educator can profitably use without knowledge of its derivation, and that the formulas printed in the textbooks constitute a dependable machine into which data may be fed and from which conclusions, even discoveries of vast moment for human welfare, will eventuate automatically. Two boys on the top of a Fifth Avenue bus arrested my attention with a scrap of conversation. Said the first, "But I don't understand it"; whereupon his companion replied, "Understand it? Gosh, man, why should you try to? It's a formula!" The world is full of men who want to take formulas on faith, arguing that they can utilize a technique whose basis they do not understand, exactly as they drive an automobile which they could not take to pieces and reconstruct. The analogy has something to recommend it. Certainly, most of the computation and tabulation called for in a statistical study can be done by clerks who merely follow directions. For the man who is directing research, however, the analogy fails. In driving a car we have a perfect and obvious check upon the success with which gears and steering wheel are managed. In choosing a statistical procedure, no such obvious check is available. A formula is always based upon assumptions made during the process of derivation and these assumptions limit its application. The formulas printed in texts are often special cases of longer ones and deduced from them by the application of very special assumptions. The choice of a different formula may vitally affect the nature of one's results, but there is usually nothing in the results themselves to validate the method by which they were reached.

The Nature of Statistical Inference. Statistical reasoning differs from mathematical reasoning in another important way. For the mathematician, conclusions follow inexorably and inevitably from premises; his reasoning eventuates in a single immutable law which is always invariably true when the original conditions prevail. The statistician derives no law which can apply invariably to all members of the population which he studies; but he deduces trends, tendencies, which are true in the main for the group, but which may not hold at all for a given individual. He speaks of the central tendency of the group and of the tendency of the group to depart therefrom, of the scattering or divergence of the group from that central tendency. The mathematician knows all his premises; the statistician can usually measure only part of the influences which play upon the subjects he is studying. The statistician is usually working in a field where events are brought about by a highly complicated plexus of causes only part of which can be measured, and therefore he speaks of probability rather than certainty. He recognizes that when he is studying one hundred cases and is attempting to generalize for ten thousand cases the results which he obtained from the hundred, then every measure he has computed for the hundred probably differs a little from what he would find if he computed the same measure for the ten thousand. Such sampling errors cannot possibly be avoided and can be mitigated only by the use of larger samples. When a mathematician speaks of an error, he means a mistake. When a statistician speaks of a sampling error, or when he computes a probable error in the attempt to measure the significance of his sampling error, he is not dealing with a mistake but with a fundamental characteristic of the nature of the universe which makes one sample differ slightly from another.

It is the very essence of statistical method that it describes the trends and the general characteristics of populations, but that these tendencies cannot be asserted as necessarily valid for each of the individuals which constitute the group. To describe these tendencies and relations with objectivity and precision, quantitative and numerical measures are naturally called for. If we say, "Most of the teachers in the country schools of America receive very low salaries," "Girls have a keener perception of color than boys," "Stuttering has a tendency to be associated with left-handedness," or "An increasing number of industrial concerns refuse to hire applicants who are over forty years of age," we are making statements which, true or false, are essentially statistical in nature because they are attempting to describe general characteristics of populations—but they are vague. Numerical methods of description would have to be employed before these statements could be rendered precise, and as soon as that is done we are beginning to use the statistical method, even if it is only in its most rudimentary stages.

II. Mathematical Training for Statistical Work

A Dilemma. We have already seen that statistical method is rapidly becoming the language in which much of the research in economics, business, psychology, education, finance, life insurance, industry, anthropometry, biology, medicine, and government is being cast. More and more an understanding of statistical terminology becomes necessary in order to read the technical literature of research, and this is true even of such subjects as religious education, vocational guidance, and child welfare. We have already seen that neither the man trained solely in mathematical theory nor the man ignorant of mathematics does the best work in organizing statistical inquiries. This recent and rapidly developing general interest in statistical studies has created a dilemma for which there is no easy solution. In some fields, such as education and psychology, it often seems that an understanding of statistical methods is almost a requisite for successful research; yet many of the keenest minds in these fields do not have the mathematical preparation which makes any thorough study of statistics feasible. Men who have a high degree of scholarship in their own field are finding themselves hampered in undertaking some piece of research because it demands statistical treatment; other men find an increasing proportion of the literature of their field unintelligible because it is phrased in an unfamiliar statistical language; still others, deciding to study statistical method, discover that they have unfortunately forgotten the algebra and the arithmetic which they had supposed they would never need again. The average man—a phrase which is itself a statistical abstraction rather than a description of a real person—finds that to a certain degree he must think statistically in order to read his newspaper and the current magazines.

Suppose, for example, that a nutrition expert interested in the feeding of small children wants to find out whether two-year-olds have their appetites stimulated by the color of the food which is offered them, and plans to make daily observations of fifty nursery school children. His training is in child psychology and food chemistry, but as the investigation progresses he finds himself involved in a statistical study of some complexity. Or again, a teacher of physical education wants to arrive at an index of physical capacity which will allow him to estimate in advance the ease with which a given high school student can master a particular sport, say swimming. His interests are in anatomy and physical training, while this problem is largely one of applied mathematics.

In his introduction to Wood's *Measurement in Higher Education* (1923), Professor Louis Terman says:

> From the language of statistics there is no escape if we wish to go beyond the limits of personal opinion and individual bias. Worthwhile evaluations in higher education will continue to be as rare as they now unhappily are until the rank and file of college and university teachers become able to think in more exact quantitative terms than they are yet accustomed to.

More than a quarter of a century ago H. G. Wells said:

> The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity and a means of expression more exact, compact, and ready than ordinary language. The great body of physical science, a great deal of the essential facts of financial science, and endless social and political problems are only accessible and only thinkable to those who have had a sound training in mathematical analysis, and the time may not be far remote when it will be understood that for complete initiation as an efficient citizen of one of the great new complex world-wide states that are now developing it is as necessary to be able to compute, to think in averages and maxima and minima as it is now to be able to read and to write. (*Mankind in the Making*, 1904, pp. 191-192.)

The time of which Wells then spoke is now imminent.

Mathematical Preparation. How much mathematics should one know before undertaking the study of statistics? No authoritative answer can be given to this question, for no one, so far as the writer knows, has made a careful, unprejudiced analysis to see what mathematical knowledge is needed for various statistical undertakings. If a canvass of expert opinions were made, they would undoubtedly range all the way from that of the mathematician who would insist upon a doctorate in pure mathematics to that of the man who once told the writer that he considered much study of mathematics an actual detriment, because he thought a statistician was freer and more original if he did not know too much about mathematics. It is obvious that there are many levels at which statistical work is carried on, and it would be valuable to have a thorough study made of the optimum mathematical preparation for each level. The suggestions which follow must be interpreted as based solely on the personal opinion of the writer, which in turn is derived from the observation of students and from the results of a statistical study of the preparation and accomplishment of over four hundred students of elementary statistical method.

The clerical worker who merely tabulates and copies raw data probably performs no mathematical function at all. The computer who works under the close supervision of someone else needs to have a flair for figures, skill on computing machines, a high sense of accuracy and reliability, and enough knowledge of arithmetic and algebra to enable him to see short cuts in arithmetic operations, but he can be a competent worker on this level with relatively little theoretical training. He can make extensive computations under direction without much understanding of the import of his work.

The student who hopes to do anything at all with the theory of statistics should have, as minimum preparation, differential calculus. While it is true that Yule wrote his *Introduction to the Theory of Statistics* without any of the notation of the calculus, nevertheless he could not avoid its general method, and most of his readers will agree that he did not succeed in simplifying his material by this expedient. The student who goes beyond the first stages in his study of statistical theory and who attempts to read the original memoirs in which important derivations are set forth, will find that he needs to know integral calculus, differential equations, theory of probability, a great deal about the convergence of series, function theory—in fact, almost any form of mathematical analysis which he has studied will be of ultimate use. The geometries are in general less pertinent, although there are one or two important papers which have utilized geometric concepts. The man who is to do original research in statistical theory will be thankful for all the mathematical analysis he has studied, and will be exceptional if he does not feel the urgent need of more training than he has had in mathematics.

Between these extremes, the clerical worker and the student of theoretical statistics, there is a large and rapidly growing group whose needs are more difficult to define and to meet. These are the men with primary interest in other fields who need some knowledge of statistical method in order to read the technical literature of their field, to interpret and evaluate the research of their fellows, and to organize data derived from studies of their own. They are often impatient with any attempt to teach them statistical theory, and they say they are interested "only" in practical interpretation and critical evaluation, failing to understand that critical evaluation and wise interpretation often call for a fine combination of acumen, wide knowledge of the field of study, and some knowledge of statistical theory. The man who is reduced to quotation of what other people have said or written about a formula, having no first-hand knowledge of its bases, may also be limited to imitation and quotation when he attempts to interpret its practical meaning in a concrete case.

However, let us suppose that we are attempting to teach as much of statistical method as can be compassed without the derivation of formulas, teaching only statistical computation and as much critical interpretation as is intellectually feasible to students who make no mathematical derivations. What mathematics is essential to such a program? If we postulate a course in statistics stripped to a minimum of mathematical content and designed to be of the utmost practical help to the person who has studied no mathematics beyond the high school, what topics in secondary school mathematics will be most needed? Here again it is necessary to make a declaimer and to admit that there is no authority for an answer save personal opinion based on a study of the needs and difficulties of a good many mature students suffering from mathematical anemia.

The question falls into two parts: Which of the topics now commonly taught in the secondary schools do students of elementary statistics use most? What topics not commonly taught in secondary school mathematics would be useful to the prospective student of statistics, simple enough to be grasped by high school pupils, and of sufficient utility for the social sciences to merit consideration as possible additions to an already crowded curriculum?

III. Topics in Arithmetic and Algebra Most Needed for Beginning Statistics

Attitudes and Habits. For success in the general processes of elementary statistical method, it appears that mathematical information and specific skills are less important than certain attitudes of mind which are sometimes regarded as by-products. That elusive thing which we call mathematical ability seems to be more essential than mathematical training unless the training produces these habits of mind. If a student has forgotten how to handle radicals and logarithms he can relearn these techniques easily. If he has never rightly understood the import of a formula, if he never knew what the solution of equations or the reduction of fractions were about but merely acquired skill in performing certain tricks which produced an answer, if he thinks variable and unknown to be synonymous terms, if he has never seen arithmetic generalized into algebra, these are matters much more serious than a total lapse of memory. Worst of all, if he is unable to think in terms of symbolism, is frightened by algebraic formulas, panic-stricken when obliged to compute, and without conscience in the matter of accuracy, he has a heavy load of old habits to discard before he can hope for any progress in statistical studies. Fortunately, the ways of thinking which the statistician would urge the mathematics teacher to inculcate and develop are ways of thinking which mathematicians also value highly. The following are of central importance:

1. *Ability to Think in Terms of Symbolism.* The language of statistical theory and method is highly symbolic, and no other single ability seems so closely related to success in this field as the ability to read meaning directly from symbolism. In a group of prognostic tests which we have been giving to students of elementary statistics at the beginning of the first term's work, a short symbolism test including only eleven items shows a correlation of .55 with marks at the end of the first term. For so short a test this is remarkable. When the symbolism test is made longer, the correlation will undoubtedly be still higher. Of all the other prognostic tests with which we have been experimenting, none, not even a standardized test of general intelligence, shows so close a re-

lationship to the term's marks. Even for the pupil who will never study statistics, this ability to use symbolism as a language in which to express his ideas is one of the most readily defensible aims for the teaching of algebra, though not one of the most easily realized. Teachers of high school mathematics can scarcely put too much emphasis upon translation from symbols to words and from words to symbols. In addition to practice in these two forms of translation there should be practice in expressing in symbolic form the pupil's own ideas about quantitative matters, a sort of free symbolic composition. Ninth grade pupils enjoy this when the ability has been developed by carefully graded exercises. It would be of inestimable value to those who will some day study statistical method.

2. *Correct Thinking About Variables.* Hazy thinking which permits a pupil to confuse variables with unknowns because both are often represented by the letters x and y may not be inconsistent with high marks in a high school algebra course, where it is often possible to achieve correct answers blindly by merely following the rules of the game; but such confusion is a very serious handicap when algebra is to be applied to statistical method. For example, suppose we let x_1 represent the height in inches of one boy, x_2 the height of a second, and so on, x_n being the height of the nth boy. Then the sum of all the heights divided by the number of boys will be the mean (or average) height, or $M_H = \frac{\Sigma x}{N}$. Now clearly x is not an unknown here, for we are not trying to find the value of some missing number, but it is a variable representing a class of numbers to all members of which the formula refers.

In arithmetic a symbol is always associated with the same number, 4 having always the same meaning no matter where it occurs. In algebra the pupil early discovers that x or n (or any other letter) may have one value in one problem and another value in another problem; but so long as he is solving such an equation as $3x + 2 = 20$, x has only one value for that equation. He sees readily enough that x may mean 6 in this equation while in another it may mean 4 or 7 or something else, but he is still essentially on the arithmetic level because during the discussion one symbol stands for one number, that number being temporarily unknown. To give the pupil a concept of variables is psychologically more difficult as well as more stimulating and ultimately more important. This is one of the most valuable contributions algebra makes to human thinking and ought to be approached with care and thoroughness. The enormous power of algebra is largely inherent in the fact that a single symbol can be used to represent every one of a class of numbers. Almost any bright pupil can learn to manipulate formulas mechanically, to evaluate the formula by substituting given values of the variables, and to change the subject of the formula; but unless he has grasped the meaning of a variable he cannot think properly about formulas, cannot compose formulas to express relations, cannot have any idea of functionality. Unfortunately, many pupils can write glibly $13n + 7n = 20n$ without realizing that this means, "If seven times *any number* is added to thirteen times that same number, the result will always be twenty times the original number." If anyone doubts this statement it is only necessary to ask a class to find the value of $13 \times 43 + 7 \times 43$, and to hand in all their scratch work, and then to note how few think of multiplying 43×20 and how many make the two separate multiplications and add the results.

In statistics we deal constantly with variables, while only seldom do we solve equations to discover the value of unknowns, and the student who has learned to think of x as standing always for a single missing number has a mental handicap to overcome.

3. *Freedom from Fear.* Among mature, educated men and women, graduate students with intelligence well above average, there exists a surprising amount of fear of anything savoring of arithmetic or algebra, old inhibitions which are rooted in arithmetic failures and worries in the early grades, old strains and anxieties which can often be traced back to a teacher who was scornful when answers did not come out right, or who tried to hurry children beyond their capabilities. It is usually a revelation to the student to discover that a considerable part of the fear and worry which he had been attributing to the difficulties of statistics actually had their origin in early misadventures with arithmetic or algebra, and when to this discovery he adds the discovery that a little well-directed practice will rid him of his sense of mathematical inferiority, he achieves a joyous freedom. But ought any subject to have such serious emotional connotations among men and women who are otherwise sensible and intelligent? Will the boys and girls who are being taught arithmetic, algebra, and geometry to-day have to carry a similar load of emotional condi-

tioning toward mathematics? Have we even yet learned to adapt our teaching to the pupil so that he may work up to his individual capacity without being constantly humiliated and scarred by failure to do successfully work which is beyond his capacity? Can we learn to teach so that boys and girls may have the triumph of becoming mathematical adventurers and discoverers and so secure a self-confidence which will be a positive force in building their personalities?

4. *Right Attitudes Toward the Outcome of Computation.* The secondary school pupil usually considers his computations, his algebraic manipulation, his geometric reasoning, and his trigonometric analysis vindicated by the approval of a teacher or by agreement with published answers. Methods of checking do not interest him much; they seem like an unnecessary labor imposed by an exacting taskmaster when comparison with an answer known to be right offers a more direct proof of his work. In statistical investigations there is no such authority against which one may measure his work, and methods of checking become of paramount importance. *The habit of checking each step of a computation* or of finding two independent ways to reach the same result must be developed in any one who hopes to do valuable statistical work.

The habit of estimating in advance of computation what is a probable value for its outcome and of checking each computed value by common sense to see if it is reasonable will save the statistician much grief. Also, the general value of these habits makes them worth considerable attention from the teacher of mathematics.

A conscience about accuracy is necessary to the good statistician. The correctness of his work is ordinarily taken for granted by his readers, and only rarely does one man recompute the measures published by another. His work must stand by itself and he must be able to vouch for its correctness. He should be sensitive about the accuracy of his computations, of his tabulations, of the measurements from which his data were derived. He should recognize that the number of decimal places which he carries in his final results is a tacit pledge of the degree of accuracy of his original measurements, and that he should carry these results only so far as is appropriate to the precision of the original measurements. He must always realize that unless his problem is a trivial one, the outcome of his computations may provide information on which will be based decisions of importance for human welfare, and that therefore he has no right to offer any but trustworthy work.

What training can we give a high school student to develop in him this sense of responsibility? Answers are available for checking; why bother about excessive accuracy? No social consequences wait upon the outcome of the usual algebra problem; a slip in the progress of a geometric proof can bring disaster to no one unless it be to the perpetrator of the slip. Certainly a desire for accuracy does not arrive as the result of verbal argument on the part of the teacher.

Probably the best expedient is to make one pupil or a small group of pupils responsible for securing data and computing the results in some matter about which the class wishes to have information. This may approximate the situation of the statistician, whose work has important social consequences and who is stimulated by the thought that the outcome may remain unknown unless he finds it.

Information and Skills. For a course of the type we are now postulating the necessary mathematical techniques are very simple—skill and a degree of rapidity in computation, knowledge of arithmetic short cuts, ability to place a decimal point, to take square root, to read a mathematical table, to change the subject of a formula, to evaluate a formula, to transform fractions, to operate with complex fractions, to plot points on coördinate axes, to make statistical graphs and to interpret them, to draw the graph of a linear equation and to know the import of the slope of a line, and to handle radicals. The use of a slide rule, of computing machines and of logarithms is highly desirable, as is also the ability to interpolate.

The formula given below indicates the complexity of structure to be encountered in computing a coefficient of correlation:

$$\frac{\dfrac{a}{N} - \dfrac{bc}{N^2}}{\sqrt{\dfrac{d}{N} - \dfrac{b^2}{N^2}}\sqrt{\dfrac{e}{N} - \dfrac{c^2}{N^2}}}.$$

One should be able to change the form of this fraction as convenient and to know enough about radicals to deal with the denominator. Again, it is necessary to evaluate expressions of the forms

$$\sqrt{1-(1-a^2)(1-b^2)(1-c^2)},$$

$$a\frac{b\sqrt{1-c^2}\sqrt{1-d^2}}{c\sqrt{1-f^2}\sqrt{1-g^2}}, \quad \text{and} \quad \frac{a-bc}{\sqrt{1-b^2}\sqrt{1-c^2}}.$$

These suggest about the limit of algebraic difficulty likely to be encountered in a first-year course in which formulas are not derived.

IV. Statistical Method for High School Students

A Challenge to Teachers. Any one vitally concerned with the teaching of high school pupils and observant of the rapidly growing public need for some knowledge of quantitative method in social problems must be asking what portions of statistical method can be brought within the comprehension of high school boys and girls, and in what way these can best be presented to them. If some aspects of statistical method are to be taught in high school, shall this be done by the mathematics teachers or by the social science teachers? Shall a new course be created, shall a new unit be added to the social science work, or a new unit be added to the work in mathematics? Shall it be required or elective, for seniors or underclassmen?

These questions call for much study and creative teaching on the part of high school teachers with pioneering spirit, and there seems every reason to expect that the next decade may produce significant changes in the program of both high schools and colleges. The situation is full of challenge for those teachers of high school mathematics who like to leave the beaten path and adventure a bit, who are not afraid of the hard study necessary to prepare themselves for teaching in a new field, and who have a genuine interest in that type of social problem which can be approached by a quantitative study of mass phenomena. Such teachers will need first to make themselves thoroughly at home in statistical method, not merely with its elementary phases but with its spirit and some of its theory. It will be most unfortunate if teachers who have had only a six-weeks' summer course in statistical method are the ones who undertake this pioneering, because the selection of material for a simple course is not in itself a simple task, and cannot be well done by the person whose knowledge is elementary.

Suggested Materials. As a starting point for creative and experimental work in organizing units in statistics to be added to high school mathematics courses, the writer suggests a number of topics which she has taught to ninth grade pupils of average ability who found the material interesting, stimulating, and no more difficult than the rest of their ninth grade mathematics.

1. *Graphs.* Almost any good junior high school text now contains valuable material on the statistical as well as the mathematical graph. Special emphasis should be placed on the *criticism of published graphs*, the analysis of their strong and weak points, and suggestions of alternate ways in which the same material might be presented, with the advantages and disadvantages of each form. Most of the modern junior high school texts now include a treatment of the histogram and frequency curve. From these, the *cumulative frequency curve* follows easily. If the raw frequencies are turned into per cents and two or more distributions are plotted on the same axes, the resulting diagram provides a way of comparing two groups which reveals at once many things not discernible from the original distributions. Such diagrams may be used to compare the work of two sections of a class on the same test, to compare the work of a class with published norms for a standardized test, to compare scores made by boys with those made by girls, or to compare the test scores made by a class at the beginning of a term with the scores of the same class on the same test at a later date. All of these comparisons are matters of genuine interest to the class.

For later statistical work, it is valuable to know how *to find the equation for a given line* and this is not necessarily a task so difficult that it must be reserved for college courses in analytic geometry. Incidentally, this problem has as much intrinsic mathematical interest as its converse which is commonly taught, and it can be treated in a manner simple enough for ninth grade pupils. The pupils may be given a practical problem in measuring any two variables that have a linear relationship and plotting the resulting pairs of measures on coördinate axes. Because of slight errors of measurement—unavoidable inaccuracies which are due to the fallibility of human eyes and hands and measuring instruments—the resulting pairs of measures will cluster about a straight line without actually falling upon it. After the data have been plotted, the pupil should draw the straight line which he thinks is the best fit for this swarm of points. Then he should find the equation for this line of best fit. Such a problem may be used to open up a general method of deriving laws which express tendencies of physical or social data, to suggest the general method of curve fitting as employed in the various sciences, to extend and enrich the pupil's understanding of graphs, and to introduce the idea of errors of observation, a concept pregnant with intellectual challenge if treated by a teacher who has grasped its philosophical import.

2. *The Percentile System.* The simplicity of the percentile scheme (including median, deciles, quartiles, quartile deviation), its frequent use to define the standing of high school or college students on standardized tests, its wide usefulness for describing the performance of an individual in terms of his position within a group, the ease with which real problems within the comprehension of adolescents may be assembled, and the fact that the process of computing a percentile offers an attractive application of percentage, an introduction to the idea of interpolation and a simple problem in intuitive geometry, all make the percentile system admirably adapted to the end of the junior high school mathematics course. Most of the topics studied in a course in statistical method are so interwoven that no one of them can be truly understood without knowledge of many others. Because of this interdependence of subject matter there seems to be a fairly circumscribed choice of topics suitable for the secondary school. The percentile system is one of the few topics which can be studied satisfactorily without the vexation of continually needing an understanding of advanced work to clarify its meaning. This is, moreover, a field in which children can be encouraged to produce their own problems, to make measurements and to use the percentile system for reporting results to the class. This furnishes an opportunity for the pupil to get a little taste of the thrill of original research and gives him a simple language in which to report the results of his own independent labors. This is less difficult for the adolescent to understand than some of the problems in financial mathematics which the junior high school pupil is mastering, and its social uses are no less real.

3. *Averages.* The concept of central tendency, or average, is

fundamental to the way the modern man thinks about his world. We talk of the average length of life for various occupations, average mileage we get per gallon of gasoline, average size of classes, average cost of commodities, average age of men and women at marriage, average salary of high school teachers, average age of children entering high school. It is well for people to know that there is more than one method for determining central tendency, more than one kind of average, and that in some situations where the arithmetic mean is misleading one of the others may give a truer picture of the group.

The arithmetic mean is easily mastered, likewise the mode. If percentiles have already been studied, the pupils know the median and are ready now to consider the advantages and disadvantages of each of these three averages and to discuss which is the best to use in a given concrete situation. While the harmonic and geometric means are used somewhat less frequently in practice, they afford attractive illustrations of mathematical principles and provide excellent applications for work in fractions and in logarithms, and are probably not too difficult for high school work. (Because I have not tried to teach the harmonic and geometric means to high school pupils, I hesitate to make a definite recommendation.)

The discussion of averages provides an opportunity to clarify for the pupil the essential nature of statistical inquiry, to show him both the importance and the limitations of drawing information from individual cases, and also the necessity of broadening the scope of a study to take in large groups of cases in order to generalize results of observation. For the sake of a satisfactory life among his fellows he needs to see that an average is but a partial description of a group, so that he may not fall into the error of scorning individuals who deviate from that average. He can be shown the need for a measure of the variability of a group as well as a measure of its central tendency. In the percentile system he has already seen such a measure, and he may be told that there is a measure of variability which goes with the arithmetic mean just as the quartile deviation goes with the median, but that it is a little more difficult to understand and that he will have to wait for further work in statistical method to find it.

4. *Relationship.* The tendency of two traits to be associated, so that wh[illegible] of them is large the other is likely to be large

also, or when one is absent the other is likely to be absent, is constantly under discussion. Are the swiftest computers the most accurate? Does education improve moral character and increase powers of leadership? Are the more intelligent people stronger or weaker physically than the less intelligent? Does the study of Latin improve the clarity and beauty of style in English composition? What qualities are associated with success in college? Questions which, like these, depend for an answer upon a measure of the interrelationship of two traits are innumerable.

The computation of any precise measure of correlation seems to the writer entirely out of the question for high school pupils, but there are three graphic methods of presenting data of this sort which are useful, which give a rough graphic picture of the extent of relationship, and which seem within the abilities of high school pupils. The writer has proved the second one successful with ninth grade children of average intelligence. She has made no experiments with the other two for high school children, but for older students they seem at least as simple as the scatter diagram.

Space should not be taken here to describe in detail these methods of picturing relationship, and therefore the following brief paragraphs may not be clear to persons who are hearing of them for the first time. The intention is merely to suggest to those who nave studied statistical method material which is suitable for high school use. Used by a teacher who understands the material thoroughly and who has a somewhat philosoohical view of the problems of the relationship of two variate , the material suggested below may open up a new world to the students.

a) Diagram to Show Rank Correlation. Suppose that ten eighth grade pupils have taken a test in speed of computation, and their names have been set down in the rank order of their performance, Mary having made the best score, John the second, Dick the third, and so on. Suppose also that the same ten pupils have taken a test in problem solving, and that the order of their scores here is somewhat different, John now having first rank, Dick second, Bertha third, and so on. The record for the ten children is shown in the tabulation on page 130.

If there were perfect positive correlation between speed of computation and problem solving, one child would have first rank in both lists, another would have second rank in both lists, and so on, any child making the same rank in both. If there were perfect

Pupil	Rank in Computation	Rank in Problem Solving
Mary	1	4
John	2	1
Dick	3	2
Esther	4	5
Bertha	5	3
Tom	6	9
Jack	7	8
Carl	8	6
Edward	9	10
Josephine	10	7

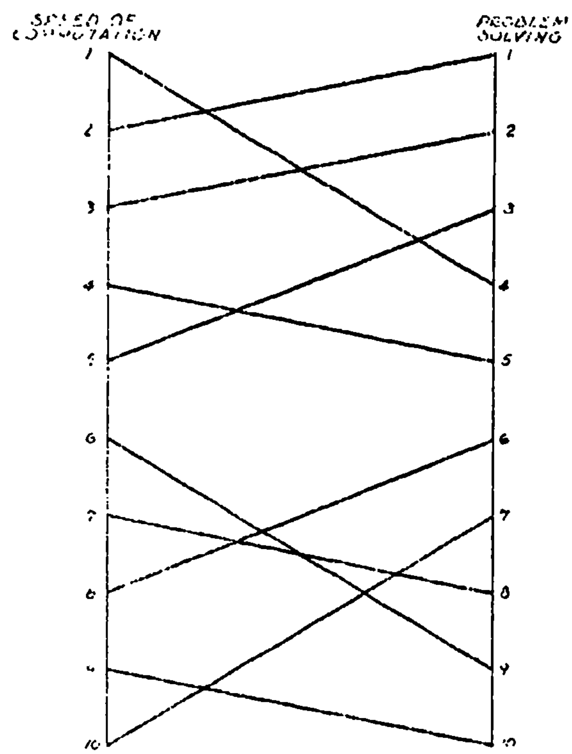
negative correlation between the two abilities, the child who stood first in one would be last in the other, and the order of excellence would be exactly reversed for the two lists. Evidently this situation shows neither perfect positive nor perfect negative relationship, but there is a general tendency for the people who are high in one trait to be high in the other, and for people who are low in one to be low in the other also. Therefore we say that the relationship is positive, though not perfect.

We will now draw two parallel lines and will lay off on each ten points equally spaced, as in the diagram on page 131. Because Mary has first rank in computation and fourth rank in problem solving, we will draw a line connecting the point 1 on the computation scale with the point 4 on the problem solving scale. Because John has second rank in computation and first rank in problem solving, we will draw a line connecting the point 2 on the computation scale with the point 1 on the problem solving scale. In a similar way lines are drawn to represent the record of each of the other pupils, one cross line representing the pair of scores for one pupil. When correlation is perfect and positive all the cross lines are parallel. The more crisscrossing there is, the lower is the correlation, and the less relationship is there between the two traits.

This form of diagram can be used to greatest advantage with a small number of cases, say less than twenty-five. The scatter diagram described later can be used for very large groups.

A class may be divided into small committees, each committee being responsible for a report on the relationship between one pair of traits, so that when the charts from all the committees are assembled they will illustrate a number of different problems, showing varying degrees of relationship, some high and some low. Unless the pupil sees a variety of such diagrams he is likely to attach too much importance to the particular shape of the one he has drawn.

Psychologically, this use of a graph to portray a statistical relationship is widely different from the graph of a mathematical function and should be attempted only by a teacher who has insight into the nature of statistical inference and who can bridge the



rather difficult gap between a mathematical function where there is a perfect correspondence between two variables and a statistical situation where the dependence is only partial. Parenthetically it may be said that the writer is convinced that it is psychologically easier for children to learn the mathematical graph first and the statistical graph later as an application of the mathematical graph, than to use the statistical graph to pave the way for the mathematical graph, as is commonly done in junior high school texts.

b) Scatter Diagram. Material of immediate interest to the class can be readily found for a problem in plotting a scatter diagram. This is an easy extension of the work in plotting points on coördinate axes and of the use of a step interval as studied in the drawing of histograms and frequency curves. It results in a swarm of points as did the experiment described in the discussion of finding the equation for a line. When the scatter diagram has been set up (see any elementary text in statistical method for the procedure), marginal frequencies should be found and the mean or average for each of the two traits should be computed. The value of each mean should then be indicated on the scale of the appropriate trait, and a line drawn across the diagram at that point. The lines of the two means then divide the area of the scatter diagram into four quadrants. The class may be asked to note the number of cases which are above the mean in both traits, below in both, or above in one and below in the other, and they may be told that when most of the cases are either above the mean for both traits or below for both, the two traits are said to show positive correlation; when most of the cases are above the mean for one trait and below it for the other, the correlation is said to be negative. This is of course a very rough statement, but ninth grade pupils understand it. They can also understand the general import of the appearance of the scatter diagram. When the correlation is high, the dots tend to cluster closely along a line; when it is low, they tend to scatter indiscriminately over the diagram.

An exceptionally mature class can go further. They can compute the mean for each vertical column in the table, marking its position by a small red dot, and then can draw the line of best fit for these red dots. In the same way they can find the mean of each horizontal row, marking its position with a small blue dot, and can draw the line of best fit for the blue dots. These two lines are called *regression lines*, and they should intersect each other at the intersection of the means of the two traits mentioned in the preceding paragraph. If we find the slope of the first line to the horizontal axis and the slope of the second line to the vertical axis, and if we multiply these two slopes together and take the square root of their product, that square root is the *coefficient of correlation*. This work would seem to be appropriate only for advanced pupils in an elective course.

c) Histograms Showing Relationship. The use of histograms to show the interrelationship of two variates may be illustrated by data gathered in our elementary statistics classes at Teachers College. At the first meeting of the class in the fall a battery of prognostic tests was given, the results from which were compared four months later with the record which the same students made in the first semester's work. Among these tests was one composed of thirty statements purporting to be algebraic identities, and the students were told to indicate which of these were true and which false.

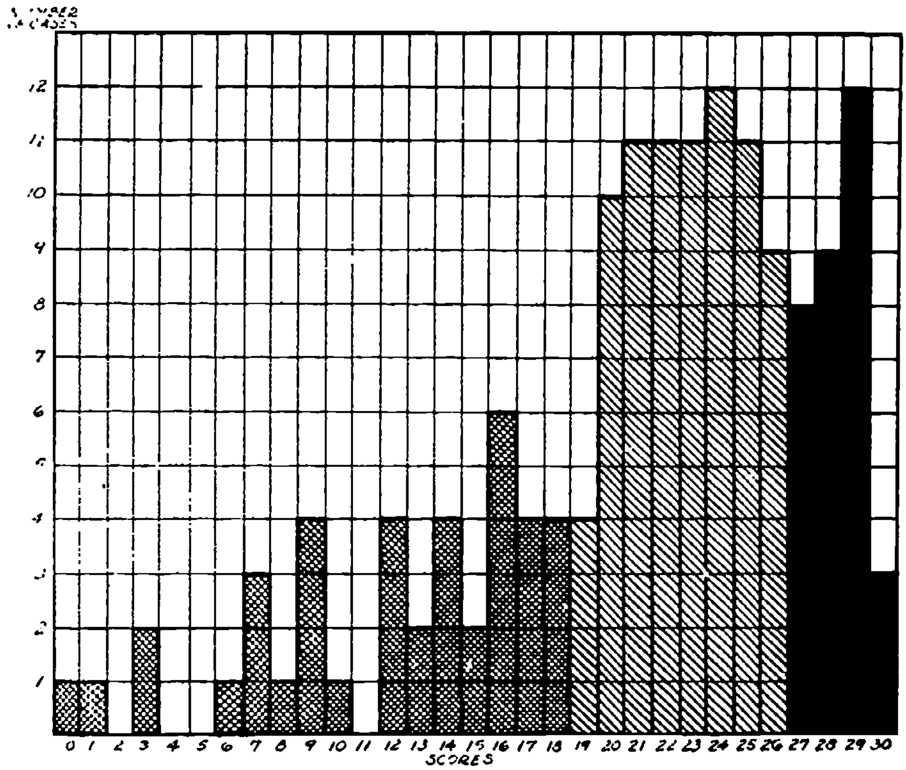


FIGURE 1. SCORES OF 151 STUDENTS IN A TRUE-FALSE TEST OF ALGEBRAIC RELATIONSHIPS, TEACHERS COLLEGE, SEPTEMBER, 1928

Black portion represents thirty-two students scoring 27 or more. Crosshatched portion represents forty students scoring 18 or less

Figure 1 is an ordinary histogram showing the distribution of scores on this true-false test, the area which represents students with scores of 27 or more being black, the area which represents students with scores of 18 or less being shaded by crosshatching. There are 32 cases in the black area, 40 in the area shaded by crosshatching, and 79 in the middle area, which is shaded by wide diagonal lines.

Figure 2 is also a histogram showing the distribution of semester grades made by these same students in the course in statistics.

These grades are stated in such a way that the average grade is approximately 50. From the original data sheet, not reproduced here, the squares on this second histogram are shaded to correspond to Figure 1. If a student had an algebra score of 27 or more, the area allotted to him in this new histogram is black. If he had a score of 18 or less, the area allotted to him is indicated by crosshatching.

A very close relationship between semester grades and algebra scores would show the black squares all at the upper end of Figure 2 and the crosshatched squares all at the lower end. A complete lack of relationship would show black, crosshatched, and wide diagonal lined squares scattered at random over the area of the polygon. Colored crayons may be used to advantage. If a more careful study is desired, the students might be numbered in the

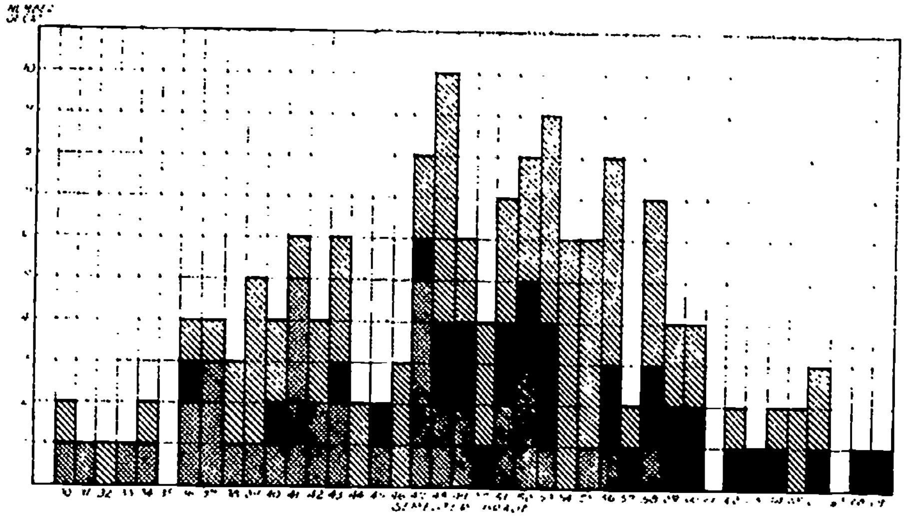


FIGURE 2. SEMESTER GRADES OF 151 STUDENTS OF ELEMENTARY STATISTICAL METHOD, TEACHERS COLLEGE, JANUARY, 1929

Black portion represents thirty-two students scoring 27 or more in the algebra test. Crosshatched section represents forty students scoring 18 or less in the algebra test

order of their scores on the algebra test, and these numbers written into the squares on both diagrams. Then it would be possible to make a case study of the student who had a high algebra score but made only 36 in semester grade and of the student with a low algebra score who came up to 58 in semester record.

Statements of the following type can be derived from this graph: Of the 22 students with lowest semester grades, only one had a high score on the algebra test. Of the 27 students with highest semester grades, only one had a very low algebra score. Of the 58 students with semester grades above 52 (more than the upper third), only four had a very low algebra score. Apparently knowledge of algebra is not a sufficient condition for high semester standing, but it seems to be almost a necessary one.

The Outlook for Instruction in Statistics. At present our graduate schools contain hundreds of students of statistics who have an inadequate background of mathematics. They struggle against unnecessary odds because in their high school and early college days no one revealed to them the vital contributions which mathematics may make to the solution of human problems. If they had seen then that "the social sciences, mathematically developed, are to be the controlling factors in civilization," as W. F. White has phrased it, they might have elected more mathematics or they might have approached the mathematics which they took with a mind-set which would make it function better when needed. If they had begun the study of statistics earlier in their educational career, there would still be time to acquire the mathematics which they need, but making the acquaintance of statistical method only after their graduate work in some other field is well advanced, they find themselves in a very difficult position.

College courses in statistical method are multiplying with great rapidity, and the number of students enrolled is multiplying still more rapidly. In all probability it will before long become customary to require an elementary course in statistical method for undergraduates who major in the social sciences—including psychology, education, and biology—just as laboratory work is now required of those who major in the physical sciences; and then it will probably become customary for a course in the mathematical theory of statistics to be considered an essential part of the work of a college department of mathematics. When these requirements are made, certain topics are almost certain to sift down into the work of the high school.

Do the teachers of high school mathematics wish to leave to the social science teachers the responsibility for instruction in quantitative methods of studying mass phenomena?

MATHEMATICS IN PHYSICS *

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Natural Philosophy. Modern high school physics takes its origin from certain courses called natural philosophy which, at least as early as 1729, began to appear in the academies of England. By the middle of the eighteenth century, they were so well established that several texts had appeared.

Natural philosophy was one of the subjects studied in the academies in this country from the first. In 1754 we find one Reverend William Smith teaching "natural and moral philosophy" at the "Publick Academy in the City of Philadelphia." [1] It was a part of the curricula of the English High in Boston (founded 1821) and of the first public high schools in New York (1825).

These courses in natural philosophy were decidedly different in character from the college physics of that time and from present-day high school courses. They were descriptive and nonmathematical even where the need for mathematical discussion was apparently clearly indicated. Largely because of this lack of quantitative treatment, the texts somewhat resembled those that might be used in some of to-day's courses in "Applied Physics." One text, Ferguson's, which enjoyed considerable popularity from about 1750 to 1825, devoted sixty-two pages to machines and forty to pumps. As Mann [2] indicates, these texts were attempting to meet the demand for secular information which the classics were unable to supply. In many instances, the authors were men whose major

* This chapter will deal with high school physics only, inasmuch as a similar treatment for college physics has already appeared. (See Congdon, A. R., *Training in High School Mathematics Essential for Success in Certain College Subjects*. Contributions to Education, No. 403. Bureau of Publications, Teachers College, Columbia University, 1930.) In this discussion, however, not only the mathematics that is necessary for success in high school physics will be discussed, but also the general function of mathematics will be emphasized.

[1] Brown, E. E., *The Making of Our Middle Schools*. Longmans, Green & Co., 1902.

[2] Mann, C. R., *The Teaching of Physics*, p. 33. The Macmillan Co., 1912.

interest lay in other directions. Very frequently they were clergymen. Their purpose was "to bring the rapidly increasing scientific knowledge of the times home to young people, without trying to force upon them that study of mathematical forms and their interrelations which was characteristic of the university physics."[3] The character of these texts and their nonmathematical nature can best be illustrated by the following quotation from the 1846 edition of *The System of Natural Philosophy*, by J. L. Comstock, a physician. (The numbers refer to sections of the original.)

> 85. If a rock is rolled from a steep mountain, its motion is at first slow and gentle, but as it proceeds downwards it moves with perpetually increased velocity, seeming to gather fresh speed every moment, until its force is such that every obstacle is overcome; trees and rocks are beat from its path, and its motion does not cease until it has rolled to a great distance on the plain.
>
> It is found by experiment that the motion of a falling body is increased, or accelerated, in regular mathematical proportions. . . . It has been ascertained by experiment, that a body, freely falling, and without resistance, passes through a distance of sixteen feet and one inch during the first second of time. Leaving out the inch, which is not necessary for our present purpose, the ratio of descent is as follows. . . .
>
> 90. If the height through which the body falls in one second be known, the height through which it falls in any proposed time may be computed. For since the height is proportional to the square of the time, the height through which it will fall in *two* seconds will be *four* times that which it falls through in *one* second. In *three* seconds it will fall through *nine* times that space; in *four* seconds *sixteen* times that of the first second; in *five* seconds, *twenty-five* times, and so on, in this proportion.

Just how far modern physics has departed from the spirit of such writing may be disclosed by a glance at the portion of any present-day text dealing with this same topic. The factors chiefly responsible for the character of books such as Comstock's were at least three in number:

1. The increasingly rapid introduction of machines into all branches of industry with the accompanying demand for more information about these devices.
2. The refusal by the colleges to accept natural philosophy as a fit subject for college entrance requirements.
3. The belated survival of the naïve, philosophic, nonexperimental point of view of mediaeval science.

[3] Mann, C. R., *op. cit.*

The Quantitative Nature of Modern High School Physics. It is beyond the scope of this chapter to discuss in detail all the influences that have determined the character of present-day high school physics. However, any attempt to get a perspective of the subject which failed to take into account the influence of the colleges would, indeed, be incomplete. This influence was felt in many directions, and is concealed in many factors which, apparently independent of the university influence, exercised their effect upon the developing subject of secondary school physics. In 1872, by recognizing physics as a subject suitable for college entrance credit, the colleges hastened the disappearance of natural philosophy from secondary school curricula. There followed a period in which the dominance of the higher institutions of learning was undisputed. They had set the stamp of their approval upon the new subject and insured its vigorous growth. What was more natural than for the subject of physics to acknowledge its fealty and to plan its courses to be as much like those of its sponsor as possible? Since the model was the college physics course, a very great emphasis was placed upon the standardization of subject matter to this pattern, to the mathematical side of the work, and to a general utilization of physics for its disciplinary values. The inevitable followed. Enrollment in physics dropped from about 23 per cent of all high school pupils in 1895, to 14 per cent in 1915, and to 9 per cent in 1922. Some of this drop must be discounted as due to the removal of the subject from required lists and to the increased diversification of the high school offering. However, making all possible allowance for such factors, it is quite evident that there has been a real falling off in enrollment in the subject, in spite of a strong reorganization movement, which began about 1905, and was featured by the report of the science committee appointed by the National Education Association.

The Movement Against Mathematics. One of the features of this reorganization has been the assault upon the mathematical portions of the subject. There has unquestionably been too great an emphasis upon this feature of the work. So long as the disciplinary theory of education held, the place of mathematics was clearly indicated. The more difficult and rigorous the course, the greater the disciplinary value gained by struggling through it. Therefore educators made the subject more stringent by inserting great numbers of mathematical problems. Tradition was with them; their model, college physics, had been featured by rigorous mathematical discipline for a long time; problems requiring mathematical solutions are the easiest to devise and correct; to require a class to solve several hundred problems was an assignment that was clear-cut and definite, and what is more, was easy to check for complete performance. Physics too frequently degenerated into a meaningless juggling of algebraic formulas, devoid of significance for either physics or mathematics. A law would be studied and then "proved"—as though it could be—by means of a few readings taken in the laboratory (very likely judiciously manipulated by the budding scientist in an attempt to make the "proof" more satisfactory).* Then would come the deluge—the solution of large numbers of problems based on the law and using the particular formula, which in some mysterious fashion was a shorthand expression for the law.

A reaction was inevitable and, as often happens with such phenomena, it went too far. It was advocated that physics should be stripped of its mathematics and made into a subject almost entirely descriptive. Thus Michelson, one of America's foremost physicists, proposes "for discussion the feasibility of a plan for the teaching of physics which avoids as far as possible the use of mathematics of even the most elementary kind, and which gives to the science of measurement only a secondary importance." Adams[4] suggests, "Mathematics should be used very little in the class in physics except for the solution of problems which are introduced in connection with the laboratory work to generalize and establish laws from given data."

Opinions such as these were widely circulated in the professional literature of a few years ago. It seemed necessary that every high school physics teacher take a stand either for or against the demathematization of his subject. If it had not been for the influence of the colleges, it is possible that a form of physics similar to the old natural philosophy might have gained a foothold. As it is, the force of the movement is manifest in the perpetuation from year to year of various hybrid courses in Household Physics, Applied Physics, Physics for Non-College Students, and the like. More recently came a swing in the other direction. The cry went

* Einstein is reported to have said, "No amount of experimentation can ever prove me right. A single experiment may at any time prove me wrong."

[4] Adams, J. W., *Correlation Between Mathematics and Physics in American High Schools.* Master's thesis, Teachers College, Columbia University, 1902.

up that the basic outline for the physics course should be the same for all. modifications in specific content being made to fit the needs of individual classes. Above all, "demathematization must stop." [5]

As we look back over the contested issues from our vantage point in the year 1930, we realize that much of the time and energy devoted to the discussion of this problem has been wasted. Once again we have been the victims of our educational myopia which renders us unable "to see the forest for the trees." It is as though a convention of carpenters should become engaged in a heated controversy over the advisability of discarding the hammer in future building construction, when no superior substitute is in sight. The question is not whether they should, or should not, employ the hammer, but rather, how means can be devised to instruct members in the more efficient use of all the tools of their trade.

So it is in physics. There is no question of whether we shall curtail the use of mathematics as much as possible or expand its use in all directions. To debate the question is to distort and alter the whole problem. Such discussion predicates a physics course, a main objective for which is: To show how the science of physics may be used to illustrate mathematical processes. Our problem is to devise means to employ more effectively this tool of the scientist's trade.

The Rôle of Mathematics in High School Physics. Rusk [6] says in this connection:

> Those who are crying for secondary school physics to throw off the burden of mathematics and become descriptive, should carefully reconsider their position and what they mean by descriptive. Mathematics should certainly not be loaded on the high school physics pupils as a burden, but without the adequate use of mathematical forms neither methods nor appreciation of precise thinking about physical phenomena can be developed. What is needed to-day is not an attempt to develop embryo mathematical physicists, but a more frequent use of simple mathematical forms by all. Even in elementary physics the pupil should be led to look upon the mathematics he uses as either simplifying the subject and making it more intelligible, or as making it directly applicable and useful. More mathematics than the pupil can thus consciously assimilate is useless and confusing.

Rusk is here suggesting certain specific aspects of the funda-

[5] Randall, D. P. and Others, "The Place of the Numerical Problem in High School Physics," *School Review*, Vol. 26: 39-43, 1918.

[6] Rusk, Rogers D., *How to Teach Physics*, p. 57. J. B. Lippincott Co., 1923.

mental concept of mathematics as a tool. To these we can add certain of our own, so that the list now stands as follows:

Mathematics can be used in connection with high school physics:

1. To simplify the subject and make it more intelligible.
2. To make it directly applicable and useful.
3. To enrich the concepts of physics.
4. To show the interrelations of the various divisions of the subject matter.

(The various items in the preceding list are not mutually exclusive. A single mathematical operation may be used in more than one of the above connections.)

Illustrations of these uses will be brought out in a later section of this chapter.

But is this a conception of the rôle of mathematics in physics that is at all new? We have already seen how it was of use when physics was taught for its disciplinary values. Let us look at mathematics used in a different fashion by the mathematical research physicists before we decide.

Many years ago, Nichols and Franklin in a preface to a text in physics said: "Calculus is the natural language of physics." Others have expanded the statement to include all mathematics.

A slightly different point of view is shown by the statement of another author, when he says, "The finished form of all science is mathematics."

With the recent discoveries in physics, the relationships between the frontier physics and its language, mathematics, have subtly altered. Mathematics is often no longer the language—it is the speaker itself. The scientific phenomena which mathematics has interpreted have been replaced by mathematical formulas which are probably not capable of being translated into any sort of mechanical model. Indeed, we are warned against making the attempt.

> This new variety of space, Einstein makes no attempt to visualize. Its definition is strictly and severely mathematical. . . . In such a space Einstein has found it possible by means of the calculus of tensors to build up a self-consistent geometry; and in terms of such a space he has formulated a general mathematical theory which as one special case reduces to Maxwell's equations, and as another to the equations of Einstein's gravitational theory.[7]

I do not maintain that this substitution of mathematical formulas for what we have been pleased to call physical reality is

[7] Heyl, P. R., *New Frontiers of Physics*, p. 135. D Appleton & Co., 1930.

universal in modern physics. I do maintain that it indicates a changed relationship.

The rôle of mathematics in secondary school physics, at the present time, is probably midway between the two positions; the one which it occupied in early high school physics in which it was used as a taskmaster to make the subject difficult, worthy of inclusion on lists of subjects suitable for college entrance, and hence of great disciplinary value; the other which it occupies to-day in research physics. To our question, "Is this a new conception of the rôle of mathematics in physics?" we are forced to give a qualified negative. The rôle of mathematics as interpreter and simplifier is not new. It has simply taken on a greatly increased significance in high school physics of to-day.

An Integrated Physics Course. Before specifically illustrating ways in which mathematics enriches high school physics, it will be necessary to develop briefly the point of view of the latter subject.

It has been evident for some time that one of the obstacles standing in the way of more satisfactory student accomplishment in high school physics was the manner in which the work was segregated into five water-tight compartments—Mechanics, Heat, Sound, Light, and Electricity. Such a procedure made it difficult for the student to grasp the underlying unity of the subject and hence to tie in each day's work with the course as a whole. Dissatisfaction resulted, and poor learning was the usual outcome. It seems necessary, then, to present the various divisions of physics, or any science, as part of a larger whole, or to state it differently, to develop the entire course around some large, unifying concept. This is not a new idea, nor is the concept difficult to obtain. Space permitting, it would be rather easy to show that the "Energy Concept" is the one best suited for such a development. Mann,[8] as far back as 1912, indicated how it might be used. Numerous writers of science works intended for popular consumption have testified to its importance (Heyl, Bridgman, Luckiesh, Jeans). In spite of this fact, textbook writers have lagged in producing texts developed around the energy concept. A possible statement of the concept for the physics course might be, "Physics is the study of energy and energy transformations which are basic to the continued existence of all life and to the universe itself."

[8] Mann, C. R., *op. cit.*

The sun is pouring out its energy at an enormous rate, millions of tons of its substance being annihilated (not burned) each second. This energy travels out in all directions into space. The earth intercepts one two-billionth of it. Some of this energy is radiant heat which the earth absorbs and transforms. As a result of its absorption, the energy of the molecules of the absorbing body is increased and various effects—temperature change, expansion, change of state—result. Or by devious steps it may be stored as potential energy in coal or in mountain lakes.

Some of the sun's energy is visible light, which man controls, redirects, and analyzes, that he may realize his universe, or render his surroundings aesthetically more satisfactory.

One of the commonest forms is mechanical energy into which both heat and light energy are frequently converted. Here we study machines, the devices most commonly employed by man in effecting energy transformations. Sound is a form of energy transmitted by particles vibrating in a certain regular fashion.

Electricity, one of the most convenient forms of energy, is one into which all the others may be converted. Static electric phenomena are due to the potential energy of electrons; current phenomena, to the kinetic energy of moving electrons (or possibly nuclei, or protons). The operation of the simple cell and storage batteries, induced currents, radio, and numerous other effects are explained in terms of these moving electrons.

This, then, is the thread which ties together the various parts of physics.

Illustrations of the Use of Mathematics in Physics

As has been indicated, there should be no question of whether or not we shall demathematize high school physics, but simply one of determining how much mathematics is needed to accomplish any one of the four functions of mathematics (see page 141), and what are the best methods for securing results.

Examples of the use of mathematics for this purpose follow:

1. **Energy Changes.** A student will get certain values from a mere discussion of the possibility of change from one energy form to another. He will almost certainly get a great deal more if, at the appropriate time in the course, mathematical developments such as the following are employed:

a) Changing heat energy to mechanical energy.

EXAMPLE. *How much work can a steam engine, which is 8% efficient, perform for every pound of coal consumed?*

The question obviously involves the relationship between heat and work. Is it a fixed one? The early work of Rumford and Joule, in particular, gives us the answer. Whenever a fixed amount of work is used up in producing heat, the quantity of heat energy produced is always the same. The converse is equally true. The relationship is that 1 B.T.U.* = 778 foot pounds. Each pound of coal when burned produces about 14,000 B.T.U. of which only 8%, or about 1,120 B.T.U., is utilized. Thus the engine can do 1,120 × 778 or 871,360 foot pounds of work. For each pound of coal, a 1,000 pound weight could be raised about 900 feet!

b) Changing one form of mechanical energy into another form of mechanical energy.

EXAMPLE. *An automobile, weighing 3,200 lbs., is travelling at the rate of 30 miles per hour. What is its kinetic energy?*

The problem in this form illustrates admirably what I mean when I say that it is not a question of whether we shall or shall not diminish the amount of mathematics to be used in high school physics, but rather of the way in which we shall use it. At this point the student will have had the definition of energy as "ability to do work" and of kinetic energy as "energy due to motion." And yet almost all texts are content to give problems which merely require the student to work out values for kinetic energy, a procedure which is almost sure to be meaningless for him. It is remotely possible that by requiring the student to label his answer with the proper unit—foot pounds—the unit of work, he will obtain a fleeting impression that this kinetic energy may, in some manner, be converted into work. Let us make sure that he realizes that this transformation does take place by adding the following problem:

The automobile hits a stone wall and is brought to a stop. How great is the shock which the bumper receives, if it bends 6"?

Using our ordinary formula, K.E. $= \frac{Wv^2}{2g}$, we learn that the car has a kinetic energy of 96,800 ft. lbs. This amount of work *must* be absorbed in bringing the car to a halt. Now work is ob-

* The B.T.U. (British Thermal Unit) is the amount of heat required to raise the temperature of 1 lb. of water 1 degree F.

tained by the formula $W = F \times s$, where s is distance, and F force. Hence if we set the 96,800 equal to the expression on the right where s is ½ foot (6″), we learn that the bumper receives a shock of 193,600 lbs. Is it any wonder that under these conditions the car would probably be wrecked?

Or let us change the problem to read: *What is the braking force, if the car is brought to rest in 100 ft. by a uniform application of the brakes?* Setting our quantity of energy equal to $F \times s$ again, using $s = 100$ ft., we learn that the braking force is 968 lbs. This includes, of course, the friction of the moving parts and of the tires on the road, the effect of engine compression, as well as the actual force of the brakes.

By setting up a problem involving the kinetic energy of a descending hammer, we can see why we so easily lose our thumb nails when we clumsily place them on the board in the path of the descending hammer head, even though nothing disastrous happens when our hand is in the open where the distance over which the energy can be absorbed is large, and hence the force is small. A similar problem will help us to understand why experience has taught us to let our hands "give" with the ball, when catching a baseball barehanded.

Certainly the student will be able to see that here is energy which can, indeed *must*, be absorbed in some fashion. Better by far to leave out any mathematical consideration of kinetic energy, in other words to dematize, than not to mathematize sufficiently to bring out all that is possible from the subject.

c) *Changing mechanical energy into electrical energy.*

In heat engines the basic energy change is from the heat energy of the coal or gasoline to mechanical energy and then to electrical energy. However, engines are rated at the maximum horse power which they can continuously deliver. Hence the following problem will help to make clear the energy change involved in this principle.

EXAMPLE. *What is the maximum electrical power* (rate of producing energy) *which a generator which is 80% efficient can develop when driven by a 40 H.P. engine?*

The factor involved is the relationship between the electric unit of power, the *watt*, and the mechanical unit, the *horse power* (H.P.). We are told that 1 H.P. = 746 watts. Thus our 40 H. P.

engine could cause the generator to develop $40 \times 746 \times 80\%$ or about 24,000 watts.

We may carry the question further if we desire by asking, "*How many 40 watt lights could such a plant operate?*" The answer is obtained, of course, by simply dividing the 24,000 by 40, which gives 600.

d) Changing electrical energy into heat energy.

EXAMPLE. *How long would it take to heat a liter* (about one quart) *of water from room temperature, 20°C., to the boiling point in an electric percolator which uses 550 watts and which is 60% efficient?*

The quantities involved are the unit of electrical energy, the *watt second* or *joule*, and the unit of heat, the *calorie*, which are related by the expression, 1 joule = 0.24 calorie. A calorie is the heat required to raise the temperature of 1 gm. of water 1°C.

The energy input into the percolator is in the form of watt seconds of which only 60% are available, due to unavoidable inefficiencies of operation. The number of seconds is unknown. Let this be represented by t. Then the number of watt seconds available for transformation into heat energy is 60% of $550t$ or $330t$ watt seconds.

Now each watt second equals 0.24 calorie, so our $330t$ watt seconds is equivalent to about $79t$ calories. Putting it somewhat differently: in one second the percolator will produce 79 calories.

Let us turn our attention to the water. There are 1,000 grams (1 liter of water weighs 1,000 grams) which are heated from room temperature, 20°C., to the boiling point, 100°C., a rise of 80 degrees. Hence $1{,}000 \times 80$, or 80,000 calories, must be supplied by the percolator. Thus it will take 80,000/79 or about 1,013 seconds, which is about 17 minutes, to produce the desired result.

We could, of course, give other examples of mathematical illustrations of energy changes, the conversion of light and other radiant forms of energy being the only ones beyond the scope of the high school course.

A final energy change, most basic of all, is that by which the sun and other stars produce their energy. Jeans and others have advanced as the most tenable hypothesis that this energy is produced by the direct change, or conversion, of matter into energy, mainly light and radiant heat. As a result of this theory, we have been deluged with such statements as these: There is enough energy in a piece of coal smaller than a pea "to take the Mauretania across the Atlantic and back"; or, In a single pound of coal (or for that matter a pound of any substance) there is sufficient energy to keep "the whole British nation going for a fortnight, domestic fires, factories, trains, power stations, ships and all."[9]

But is there a fixed relationship here, as with other energy transformations? Yes, for we are told that for every gram of matter completely destroyed 9×10^{20} ergs of energy are produced. Or if we change that to more familiar units, for every pound so destroyed about 3×10^{16} foot pounds of energy result. Darrow[10] reports that the earth receives 60 tons of energy in the form of sunlight every year, or about $\frac{1}{4}$ of a pound a minute. That means that we are receiving in the form of light, radiant heat, and other forms of energy $\frac{1}{4} \times 3 \times 10^{16}$ foot pounds of energy each minute. That is about 7.5×10^{15} foot pounds per minute, which is equivalent to about 2.3×10^{11} horse power (230,000,000,000 H.P.).

All of these energy changes are made more real and vivid by the realization that a certain quantity of one kind of energy may be converted into another form and, by the proper mathematical equation, the resultant quantity of new energy computed.

2. **Definitions and Units.** I have taken up the use of mathematics to illustrate energy changes first, simply because these transformations are basic to all of high school physics. There are, however, other uses for mathematics than these. One of the most important is in clarifying and simplifying the definitions and units of the science.

a) *Work* is a unit often used. We define work as that which is accomplished when a force acts through a distance, or, as it is sometimes defined, "the overcoming of resistance." Sometimes the proviso is added that the force must be measured in the direction in which the motion takes place. Not a very clear-cut definition, possibly. But how simple it becomes when we put it in the form of a formula, $W = F \times s$, and illustrate it by such a problem as this: "*How much work do you do when you walk to the top of a stairway 10 ft. high?* Assuming your weight as 150 lbs., the answer 1,500 ft. lbs. is obtained immediately.

[9] Jeans, Sir James, *The Universe Around Us*, p. 181. The Macmillan Company, 1929.

[10] Darrow, F. L., *The New World of Physical Discovery*, p. 330. Bobbs Merrill, 1930.

We may use a sort of ***reductio ad absurdum*** method to further clarify the meaning by assigning and discussing such a problem as, "*You are pushing a lawn mower which weighs 40 lbs. How much work do you do in pushing it 10 ft.?* Many will at first multiply the weight, 40 lbs., by the distance 10 ft., getting 400 ft. lbs. for an answer. But going back to our fundamental definition of work as force times distance, we realize that the force required to push the mower is not given when we know the weight alone. Consequently the problem as stated cannot be worked. Let us carry our problem further. Suppose friction is negligibly small. Then the work done, the product of this negligibly small force and the ten feet, is itself negligibly small. In the theoretical case when there was no friction at all, no work would be done. Suppose, though, we were to carry the mower to the top of a flight of stairs ten feet high, could we then get an answer? Our formula requires us to use a force. Do we know it? Yes, it is 40 lbs., because at the earth's surface it requires a 40 lb. force to lift a mass of 40 lbs. magnitude. Our work is now 40×10 or 400 ft. lbs.

By focusing our attention on the fact that we must use a force, the formula has helped to eliminate the confusion resulting from two situations in one of which we can use the weight of the object and in the other of which we cannot.

b) Another unit, the meaning of which is clarified by a mathematical treatment, is the *watt second*, or the more convenient, larger, *kilowatt hour*. It is easy to establish the fact that the watt is a unit of power. Now, power is defined mathematically as work/time or W/t. A watt second then is a unit of power, W/t, multiplied by a unit of time, t. Dividing out the t's we get simply W. In other words, the watt second is a unit of work, or electrical energy. Similarly the more practical kilowatt hour, which is merely a larger measure, is also a unit of electrical energy.

Other instances might be given but these are probably sufficient to illustrate this important function of mathematics in high school physics.

3. **Laws and Principles.** One illustration will be sufficient, I believe, to show how mathematics helps to clarify and enrich some of the laws of physics.

One of the most important laws in physics is the so-called "Inverse-Square Law" which states that the intensity of the illumination from a given source varies directly with the strength of the source and inversely with the square of the distance from it. Stated algebraically:

$$\text{Illumination (foot candles)} = \frac{\text{Candle power}}{\text{Distance}^2}.$$

Let us compute the illumination at distances of 1, 2, and 3 ft. from a 36 candle power light. We get 36, 9, and 4 foot candles, respectively, numbers which bear the relationship of 9 to 4 to 1 to each other or 3^2 to 2^2 to 1^2. The original distances were 1, 2, and 3 feet. The illuminations are in the *inverse* order and *squared.* Certainly this more vividly illustrates the real significance of the law than a nonmathematical discussion could possibly do.

4. **Structure of Matter.** Without being inclined to demonstrate mathematically that it is all quite possible, popular writers in this field have usually been content to astound with statements such as the following, the two parts of which are apparently contradictory.

> If all the molecules in a cubic centimeter (1/16 of a cubic inch) of hydrogen gas at ordinary temperature and pressure, were placed end to end in a single line this "string of molecular beads" would extend several million miles or many times the distance between the earth and the moon. In a cubic centimeter of hydrogen at atmospheric pressure and at the temperature of melting ice there are 2.7×10^{19} molecules, each having a diameter of 2.17×10^{-8} cm.

(And then follows the part apparently irreconcilable with the former statement about the "molecular beads.")

> Less than a *millionth of the total space* [italics mine] is occupied by the hydrogen molecules under these conditions.[11]

We may very well ask how it is possible that these molecules should reach so far when placed end to end, and yet that less than a millionth of the space should be occupied by them? Small wonder that high school physics pupils reading such a statement put it down as another one of those things beyond their comprehension.

But let us see what mathematics shows. The distance the molecules will reach when placed in a line is the sum of their diameters. If we multiply the number of molecules, 2.7×10^{19}, by the diameter of each, 2.17×10^{-8} we get about 6×10^{11} cm. This is 6×10^{6} kilometers or about 3.6×10^{6} miles, that is 3,600,000

[11] Luckiesh, Matthew, *Foundations of the Universe,* p. 34. D. Van Nostrand Company, Inc., 1925.

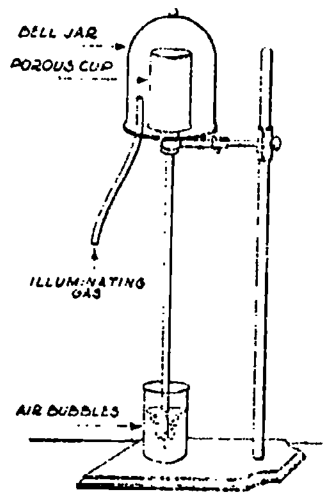
miles. Since the distance from the earth to the moon is only 240,000 miles, we see that actually the molecules would reach 15 times as far as the moon.

To check the statement about the space, we need to know the space occupied by one molecule, that is, its volume. We will need to assume that the molecule is spherical in shape, which is probably not quite true. The volume of a sphere is given by the formula $\frac{1}{6}\pi d^3$, or $\frac{1}{6} \times 3.14 \times (2.17 \times 10^{-8})^3$, or about 5.1×10^{-24} cubic centimeters. If we multiply this by the total number of molecules in one cubic centimeter, 2.7×10^{19}, we get approximately 14×10^{-5} or about 1/7,000 of a cubic centimeter. This does not agree with Luckiesh's statement. It may be he is in error. Or perhaps our fundamental assumption that the molecule is spherical is wrong. If it were in the shape of a flat, elongated ellipsoid, the volume of each molecule would be greatly reduced without affecting the length of the "molecular beads." Or we may attempt reconciliation by saying that this figure, 1/7,000, represents the portion of the space that would be occupied by *solid* spheres of the same diameter as the molecule, while it is known that the molecule is far from solid.

In any event, the fundamental picture of a volume of gas in which the molecules when placed end to end will reach vast distances, and yet in which the molecules are separated by distances which are large in comparison with the size of individual molecules, is rendered quite consistent.

One further illustration of the use of mathematics in the field of the structure of matter is afforded by the kinetic-molecular theory of gases. You will recall that this theory indicates that the molecules of gases are moving in a haphazard fashion in all directions and that when two gases are at the same temperature, the mean kinetic energy of the molecules of one gas equals the mean kinetic energy of the molecules of the second gas.

At some time during the work on heat, the fact is brought out that the molecules of the gases are moving at different speeds and that those of the lighter gases are the most rapid. Usually an experiment such as that illustrated is used to show this fact experimentally. When illuminating gas is introduced under the bell jar, its molecules being smaller and of higher speeds diffuse through the porous cup faster than the molecules of air inside escape to the outside. This produces an increased pressure inside the cup which pushes the indicating column down. On removing the jar, the reverse process takes place, a partial vacuum is produced inside the cup, and the indicating column rises to a point higher than that at which it stood originally.



Now the explanation of this result, i.e., that the molecules of illuminating gas (largely methane CH_4, carbon monoxide CO, and hydrogen H) are all lighter than air and hence move at a higher rate of speed, follows directly from the theory which asserts that the average kinetic energy of the molecules is the same. The mathematical statement is

$$\frac{w_i v_i^2}{2g} = \frac{w_a v_a^2}{2g}$$

where w_i and v_i stand for the average weight and average velocity of the molecules of various substances in illuminating gas and w_a and v_a similar quantities for air molecules. Multiplying through by $2g$ we get

$$w_i v_i^2 = w_a v_a^2.$$

Now the average value for w_i, the weight of the gas molecules, is less than for w_a. Hence in order that the equation may be preserved we realize that the molecules of illuminating gas must have a haphazard motion decidedly faster than that of the air molecules.

If we desire to make the illustration more specific, let us take the case of oxygen and hydrogen. Again we have

$$w_o v_o^2 = w_h v_h^2.$$

Now it is known that the oxygen molecule is 16 times as heavy as that of hydrogen; in other words, that $w_o = 16w_h$. Substituting this value for w_o in the preceding equation, we get

$$16w_h v_o^2 = w_h v_h^2 \quad \text{or} \quad 16v_o^2 = v_h^2.$$

Taking the square root of both sides

$$4v_o = v_h.$$

In other words, the average velocity of the hydrogen molecules is four times that of the oxygen molecules.

This is the last of four types of illustration, chosen to bring out the various uses to which the tool, mathematics, can be put in high school physics. There are, of course, other illustrations that might have been used with equal validity. Let us reiterate our fundamental belief that no one is able to hand down ex-cathedra opinions on the quantity of mathematics to be used in physics. We can only decide upon the quantity and nature of the mathematics we will employ, in assisting in the development of a certain specific unit, to perform those tasks of which mathematics is admittedly capable.

Mathematical Abilities Needed in High School Physics

Kilzer[12] has shown that, next to an interest in the subject, the outstanding factor contributing to success in physics is ability to handle the simple mathematics involved. This is merely a recent verification of something that many physics teachers have long believed. Most vigorous in their advocacy of this theory have been those teachers whom it helped to free from the responsibility for large numbers of student failures. When 25 per cent of the physics class failed (and this was not at all uncommon), the teachers could wash their hands clean of the whole affair by exclaiming, "What can you expect with the poor mathematical preparation our students get in this school?" Such an attitude is infectious and many very excellent physics teachers, at some time or other in their careers, have rallied around the banner whose slogan is, "Blame it all on the mathematics teacher."

As long as the doctrine of the transfer of training held sway, this attitude was bound to be widely accepted. To-day we realize that, whatever may be the responsibility of the mathematics department, the blame for the large number of failures that have been common in high school physics must be placed squarely upon the shoulders of one individual, the physics teacher. With this realization the saner point of view, expressed in the following quotation, has gained recognition.

> The apparent lack of transfer of training is due to both the failure to retain what was learned and the failure to see any common connecting elements between the field of mathematics and mathematics in physics. Poor work in the mathematics involved in physics is not entirely due to the

[12] Kilzer, L. R., "The Mathematics Needed in High School Physics," *School Science and Mathematics*, Vol. 29: 360-362, 1929. (An abstract of the author's Ph.D. dissertation on this subject.)

mathematics itself. . . . Thus mathematics and the understanding of the subject are dependent on ability to read and understand material connected with physics; to understand what is said by the teacher and others in the class.[13]

Before attempting to solve the problem of what to do about the acknowledged inability of students to solve numerical problems in physics, two questions need to be answered:

1. What mathematics is actually needed for the working out of the problems in physics?
2. How well can the average student handle this mathematics?

Reagan [14] solved the 241 problems found in one edition of Millikan and Gale's text, and analyzed each for the skills needed. He found that arithmetical processes were the most commonly used; addition being used 47 times; subtraction 37 times; multiplication 422 times; division 266 times; common fractions (including all processes—reduction, addition, etc.) 97 times; as well as miscellaneous skills with smaller frequencies.

Algebra was used less often, the skills with their frequencies being:

1. Translation of laws of physics into mathematical formula.....	9
2. Derivation of formula from given mathematical relationship..	6
3. Selection of formula	56
4. Solution of equation with one unknown......................	11
5. Solution of quadratic equation............................. (Not solvable by factoring.)	1
6. Squaring a binomial..	1
7. Operations with signed numbers.............................	13

Geometry was utilized with still less frequency. Only 16 theorems, of which the most important were those dealing with the similarity of triangles, proportionalities between lines, and the relation between the sides of a right triangle, were found.

The use of trigonometry, solid geometry, and the like, was not clearly indicated as necessary.

As a result of this study, Reagan concludes that the demands on mathematical ability are not unduly heavy; that the knowledge of arithmetic is satisfactory if the student can multiply and

[13] Goss, Mildred J., *Causes of Failure in High School Physics.* Contributions to Education, Vol. II. World Book Company, 1928.

[14] Reagan, G. W., "The Mathematics Involved in Solving High School Physics Problems." *School Science and Mathematics,* Vol. 25: 292-299, 1925.

divide integers up to 12 places and if he can app the laws of mensuration; that the ordinary geometry course is reasonably sure to be adequate preparation; but that algebra is the most likely to be deficient where work with formulas, ratios, and proportions is involved.

Lohr[15] prepared a 28-problem test which he administered to classes in both secondary and junior college physics. He found a rather large list of specific mathematical abilities, such as the ability to determine the volume of a sphere given the radius, to make a line graph of the relation $C. = (F. - 32)\ 5/9$, and the like, which could be solved correctly by less than 75 per cent of the members of both groups; and a somewhat smaller list, such as the subtraction of decimals, and reducing inches to centimeters, the reduction factor being given, that could be handled correctly by more than 75 per cent of the college group. The secondary group was not significantly different.

He concludes that, on the whole, "pupils come to physics with a marked ability to handle the mathematics of physics"; that the inabilities can be determined; and that it is the duty of the physics teacher to identify the mathematics difficulties, to reteach the mathematics needed, and to teach the physics of the problem situation. Note the temperateness of this last conclusion. There is no tendency to lay the blame on the mathematics teacher. Physics teachers have come to realize that be a student as well trained as you please in mathematics, unless the physics in the problem situation is well taught there will be an unsatisfactory accomplishment. The difficulties inherent in the situation, even when all these conditions have been met, is well brought out by Nyberg's[16] criticism of Lohr's work. The writer points out that some of the inabilities revealed might be due to small differences between the wording commonly employed in physics texts and that used in algebras. These small differences are very likely *not* to be explained away in teaching the physics of the problem. For example, one of the skills in which large numbers of the students were deficient was the ability to determine the per cent of error between the true and the measured scores. Nyberg points out that a student might fail to

[15] Lohr, Vergil C., "A Study of the Mathematical Abilities, Powers and Skills as Shown by Certain Classes in Physical Science." *School Science and Mathematics*, Vol. 25: 834-844, 1925.

[16] Nyberg, Jos. A., "A Discussion of an Article on Mathematical Abilities and Physics." *School Science and Mathematics*, Vol. 26: 9-15, 1926.

attempt the problem through not knowing whether to compute the per cent error on the true or the measured score, even although, when the scores are reasonably close, results are virtually the same whichever is employed. Algebra texts commonly indicate which to use as a base.

Similarly, graphing the relationship C.= (F.— 32) 5/9 might have seemed more difficult than it is intrinsically, because the units for graphing were not stipulated. In other problems, Nyberg seemed to find violations of graphing rules, hazy instructions, and unfamiliar terminology creating artificial difficulties.

All this serves to illustrate the complexity of the whole problem. Beside teaching the physics of the problem situation, the teacher, in order to secure reasonably satisfactory performance in problem solving, will probably need to reteach the mathematics involved, identifying identical elements, and should also make sure that unfamiliar phraseology or deviation from mathematical practice do not needlessly complicate matters. It is quite probable that Kilzer's suggestion that a prete t of mathematical ability be given early in the course is a good one.

What Physics Asks of Mathematics

All of the suggestions of the preceding paragraph were based upon the assumption that the mathematical training of entering students is satisfactory. Let us consider what steps may be taken to make this preparation of greater value in physics, taking Reagan's [17] analysis as a starting point for our progra of mathematical training.

1. **Algebra.** Arithmetic is not commonly taught in high school, so first we will consider certain specific suggestions with respect to algebra.

a) Formulas. An examination of recent textbooks in algebra will reveal that a great deal more work with the formula is indicated than was common in earlier books. It is very probable that the type of wo k being done in this line in the best of modern schools is amply adequate for the needs of high school physics.

More specifically, a physics teacher might suggest that ample practice be given in solving a formula such as $I = \frac{E}{R}$ for both E and R and, in general, that drill be given in solving each of the

[17] Reagan, G. W., *op. cit.*

formulas commonly encountered in physics for letters of the right-hand member. In addition, considerable practice should be given in obtaining the value of one symbol when actual values for the others are given.

Sometimes physics students are asked to examine data obtained in the laboratory or elsewhere, in an attempt to discern the relationship connecting two of the quantities involved. Sometimes the data are exceedingly simple and the relationship quite exact, as in the table at the left:

Length of Uniform Wire	Voltage Drop
25 cm.	1.5 volts
50 cm.	3.1 volts
75 cm.	4.5 volts
100 cm.	5.9 volts

Physics teachers will recognize these as data from the experiment showing the relationship between resistance of various portions of a circuit and the voltage drop around that portion. An algebra student will easily discern that there is a direct relationship between the two quantities. True, it is not quite exact, but in view of the fact that these are experimental figures and hence subject to the usual errors of experiment, a consistent mathematical relationship is definitely suggested.

Data do not always come out so nicely; as, for example, in the table at the right which gives measured values for the distance covered by a "frictionless" car, moving down a taut wire, in various time intervals.

Distance	Time
14″	1 sec.
4′ 5″ or 53″	2 sec.
10′ or 120″	3 sec.
17′ 2″ or 206″	4 sec.

Here the relationship is at once less obvious, and less exact. If the student is to notice at all the tendency for a direct proportionality to exist between the distance and the *square* of the time, he needs considerable experience with a similar type of work.

It is quite common in some high school algebra classes to ask students to write a formula that will express the relationship between the quantities in a table. The work of the two tables above suggests the desirability of including a certain number of problems in which, as the data are supposed to be the result of actual measurement (and hence subject to error) the simple mathematical relationship between the quantities may be slightly in error for some, or possibly all, of the pairs of measurements.

b) Proportion and Variation. Physics texts abound in such expressions as, "The density of air, or any gas, *varies directly* as the pressure at constant temperature." And then as a mathematical equivalent of this law some such expression as $\frac{D_1}{D_2} = \frac{P_1}{P_2}$ is given. The first time this situation arises (and it arises often), the beginning student is plunged into difficulty. In many instances he has had no experience in dealing with quantities that vary directly one with the other (although he may have handled an equivalent problem disguised under a different terminology), or if he has, a method not employing a proportion may have been used. The mathematics teacher will be of very real assistance here if the use of the subscript is taught to bring out the distinction between different quantities of the same kind. (Thus D_1, D_2, D_3 or D_a, D_b, D_c may indicate different numerical values of density.) In addition, the use of expressions such as $\frac{D_1}{D_2} = \frac{P_1}{P_2}$ to show a direct variation between two quantities, and $\frac{P_1}{P_2} = \frac{V_2}{V_1}$ to indicate an inverse variation, should be thoroughly taught and utilized in a number of situations. The fact that $\frac{D_1}{D_2} = \frac{P_1}{P_2}$, $\frac{D_1}{P_1} = \frac{D_2}{P_2}$, $\frac{D_2}{P_2} = \frac{D_1}{P_1}$, and the like, are all equally valid as expressions of the same direct relationship, as well as the corresponding equivalents of the inverse variation, such as that given above, should also be brought out and thoroughly fixed by drill.

The method for solving proportions should also be taught. In addition to the usual cross-product (product of the means equals product of the extremes) method, the various ways for first simplifying a proportion should be made functional. The following proportions will make this point clear. In each case before cross-multiplying the indicated simplification should be performed.

$\frac{36}{4} = \frac{x}{9}$	Simplify the first ratio to $\frac{9}{1}$.
$\frac{25}{x} = \frac{10}{7}$	Divide both numerators by 5, changing to the form $\frac{5}{x} = \frac{2}{7}$.
$\frac{45}{8} = \frac{x}{12}$	Divide both denominators by 4 (equivalent to multiplying both sides of the equation by 4), changing to the form $\frac{45}{2} = \frac{x}{3}$.

c) Other Operations. The other algebraic skills specifically indicated by the studies of Reagan and others are likely to be thoroughly mastered in the usual algebra course, since while they occur with a small frequency in physics problems they occupy a much more prominent place in the mathematics course. For instance, Reagan found that in the 241 problems in the Millikan and Gale text only 1 solution of a quadratic equation and only 13 operations with signed numbers were specifically required.

While not fundamental to the solution of physics problems there are several topics, not commonly included in an algebra course, which bear on the ability to handle the mathematics of physics. Taught in the algebra class, they would at once lighten the physics teacher's burden and add topics which would be of interest in algebra.

d) Metric System. Even though many algebra students may not go on to take a physics course, yet the increasing use of the metric system in everyday life amply justifies its inclusion in the mathematics course. The increasing tendency for authors of algebra texts to include a short section on the metric system, bears testimony to the value of such a unit and to its fitness in a modern, laboratory-type course. An opportunity is here afforded for correlation between the mathematics and the physics departments. Metric rules, gram weights and balances, graduated cylinders and liter measures, all can be borrowed. Where possible, the meeting place of the algebra class may be changed to the physics laboratory for the duration of this unit. The first strip-film roll from the set on Mechanics (sold by the Spencer Lens Co.), which deals with the metric system, may be shown with advantage. Not only is such correlated work helpful for the physics teacher in rendering unnecessary any detailed study for the metric system, but it is an interesting variation from the usual routine of the algebra class.

e) The Slide Rule. If we accept the theory of the rôle of mathematics put forth earlier in this article, we are forced to the conclusion that any device which simplifies the mere mechanics of multiplying, dividing, and the like, is worth while. The slide rule is such a device. We may summarize its advantages as follows:

1. By reducing the amount of attention that needs to be devoted to the arithmetical processes by which the answer is obtained, it allows the focusing of a more undivided attention upon the physical principle involved.
2. When its use is thoroughly mastered, a great deal of time is saved. A greater number of problems may be worked, or

the time spent on the mathematical development of a given topic reduced.

3. Its use prevents obtaining results with an entirely fictitious accuracy. If the measurement of the diameter of a circle gives 2.46 cm., the final answer cannot be accurate to a greater number of places. We are all too familiar with answers such as 4.75292664, which is obtained with the usual area formula, multiplied out "longhand" using $\pi = 3.1416$. In this answer, the latter decimals are of no significance whatsoever as they were all obtained by a multiplication involving the doubtful measured figure *6* of the original measurement. The greatest number of places to which we are justified in carrying the answer is three, giving 4.75 sq. cm. for the area. The slide rule more or less automatically takes care of such situations.

Not only is the slide rule of advantage in physics but the mastery of its operation is quite within the grasp of a ninth grade algebra student. In fact the manner in which multiplication and division are performed can be taught in a very few minutes.

The scales of a slide rule are logarithmic. Because of the decreasing space between successive numbers of the same order, the spacing in different parts of the rule is not the same. Between 1 and 2, each of the smallest divisions indicates a value of .01 of the whole space; between 2 and 3, and 3 and 4, .02; and from 4 to 10, the end of the rule, .05 of a whole space. This indicates the folly of stopping with a mere demonstration of the principle of the rule. We must insure an ability to interpret the scales quickly and accurately—an ability which can be acquired only after considerable drill. The algebra student who, by himself, will acquire any real ability to use the rule following a short demonstration is an exceedingly rare individual.

The Keuffel and Esser Company, 27 Fulton Street, New York City, who have rules that retail for as little as 75 cents or one dollar (subject to a school discount), also manufacture a large 8′ demonstration rule, and furnish helpful suggestions on methods for teaching the use of the rule. In my own work, I have found mimeographed work sheets helpful in this connection. On one such sheet, three drawings of the 10″ scale, full size, were shown. One showed only the major divisions, the second both major and secondary, and the third drawing some of the smallest divisions. Questions were asked about the size of the different divisions and students were required to indicate the value of certain points indicated by arrows on the drawings. On a second sheet, magnifications of portions of each of the three size divisions were drawn. Arrows were directed to various points on the drawings. By estimating the reading of each arrow, practice in using the scales, when the indicator does not coincide exactly with one of the markings, was afforded the student. Other work sheets give instructions for performing the various operations—multiplying, dividing, squaring, and extracting square root, together with a selection of problems and answers.

To do a thorough job of teaching the slide rule is not the easiest of tasks, but it gives the student a tool which is useful in many future activities. Furthermore, students like the work. In the first flush of their enthusiasm they will not multiply 2×3 except on the slide rule. It is the task of the algebra teacher to utilize this early enthusiasm to secure for the students a degree of confidence in the accuracy of their results which will insure their turning to the rule as a convenient tool, and not as a novelty or toy.

Since the slide rule is based upon logarithms, many teachers may feel that a knowledge of these is necessary for mastery of the instrument. This does not follow. In industry many persons use the rule who have only the vaguest of notions of the principles involved. That these persons are any the less accurate or rapid in their use of the rule has yet to be shown.

Criticisms of the slide rule sometimes advanced are that it is inaccurate and that it encourages careless work. The slide rule has its limitations, of course, but it is amply accurate for most computations which high school students are called upon to perform, particularly where measured quantities are involved, as in physics. By bringing out the principle that an answer obtained by using measured quantities cannot be more accurate, except in the case of an average, than the least accurate quantity employed in the computation, the value of the slide rule in automatically "rounding-off" answers will be indicated and fictitiously accurate answers will be eliminated. A discussion of significant figures can be appropriately taken up in this connection. To the charge that it encourages careless work, I need simply respond that one of the most important uses to which the slide rule is put, is that of checking results of ordinary computation. In the long run, the use of the slide rule, by facilitating the checking of results, will probably increase accuracy. Any individual cases in which carelessness has followed its use are likely to be the result of failure to impress the limitations of the instrument upon its user.

f) Exponents. Any reader of popular science articles cannot help but be impressed by the number of numerical quantities such as 2.7×10^{19} or 2.17×10^{-8} which occur. This would suggest the desirability of adding to the usual work on exponents a more detailed consideration than is commonly accorded this portion of the subject. This work should include raising to a power; extracting a root; multiplying; dividing; changing a decimal to this form; changing from this form to a decimal. If, in addition, a real feeling for the bigness of such a number as 2.7×10^{19} and for the minuteness of such a quantity as 2.17×10^{-8} is developed, a valuable service will have been rendered the student.

2. Other High School Mathematics. The investigations of Reagan and others have revealed that the small amount of geometry (16 propositions) required in high school physics is almost certain to be sufficiently functional after the usual "exposure" to the subject.

Since physics is taught somewhat more commonly in the eleventh school year, algebra and geometry constitute the usual preparation of the pupils in mathematics. This being the case, a knowledge of trigonometry cannot be expected. Knowledge of the simple functions, sine, cosine, and tangent, and also an ability to use a table of their natural values are helpful, but not necessary. This small amount of trigonometry is often learned in ninth year algebra courses.

Some advanced schools are offering courses in the elementary calculus in the senior year. With a physics group composed of seniors taking both subjects, there is an opportunity for correlating the work of the two subjects in a few instances, notably in accelerated motion. Thus, taking the derivative of the equation for the distance covered by a freely falling body in a given time, $S = \frac{1}{2}gt^2$, we get $dS/dt = gt$. But dS/dt is simply a definition of velocity, so we have $v = gt$. Similarly, ta ng the derivative of this equation, we get $dv/dt = g$. Again, dv/dt is a definition for acceleration. These processes will give a better insight into the meaning of the terms "acceleration" and "velocity."

Tyler[18] puts it (speaking of a college group), "I like to tell my students that the court will not accept $v = s/t$ as a defense in a case of over-speeding; that the automobile trap for obvious reasons uses $\frac{\Delta s}{\Delta t}$; and that our definition of derivative as speed merely completes the transition to the limit."

Of course, there are no problems in the ordinary physics text that require a knowledge of calculus for their solution.

Correlated Mathematics and Physics

There have been widely varying opinions expressed as to the amount of physics that should be introduced into a mathematics course. Young says that physics "should be taught simultaneously with mathematics throughout the four years of the course, bringing the mathematical theory and the physical application into close juxtaposition."[19] In England there is customarily a great deal of correlation between the courses. Nunn, an English writer, however, affirms that problems involving science principles to be used in the mathematics course, "must be limited to those whose solution is simply the question of the straightforward mathematics."[20] I have, I hope, made my own position clear with regard to that which I consider to be the rôle of mathematics in physics. As to the rôle of science in the mathematics class, that is beyond the scope of this article. Suffice it to say that I am perfectly willing for the mathematics courses to "help themselves" to as much science as they see fit in order to enrich the content of their subjects.

And yet, in that which I am about to say, I may seem to be inconsistent. There have been at various times enthusiasts who have insisted that a combined physics and mathematics course might be worth while. In at least one case, this took the form of combining three subjects—a year of plane geometry, the mathematics of the eleventh school year, and a year of physics into a single course to run for two years. The total number of recitations in the two years were equal to those of two "majors." It would be most unscientific for me to attempt to decide finally upon the

[18] Tyler, H. W., "Mathematics in Science." *The Mathematics Teacher*, Vol. 21: 273-279, 1928.

[19] Young, J. W. A., *The Teaching of Mathematics*. Longmans, Green and Company, 1925.

[20] Nunn, T. Percy, *The Teaching of Algebra*. Longmans, Green and Company, 1919.

merits or demerits of such a plan. It can unquestionably be carried out with a good group of students who will remain through the two years; and time will be saved. In the same way, I am sure that a group of equal ability could be prepared to take the College Entrance Examination in physics in one-half to two-thirds the usual time. Unquestionably it would be a course in which many of the larger generalizations and principles of physics would be slighted, in the hurry to obtain the necessary information and abilities (largely mathematical) necessary to make this scholastic hurdle. In the same way, I feel that many of the larger principles of physics would be lost in some of the forced correlations of this combined mathematics and physics course. Instead of any real appreciation of the energy concept and what it means to man and his universe, the student would probably gain a very real understanding of the fact that science makes excellent illustrative material for the study of mathematics.

Summary

In this chapter, I have attempted to trace briefly the development of to-day's high school physics, to show how it changed from a highly descriptive, nonmathematical subject to one which, under the domination of the colleges, and in virtue of the doctrine of discipline, became a highly mathematized subject, and how finally, during the last decade, the pendulum has swung back, going perhaps too far in its swing.

Whatever may be the final solution to this problem, it can be demonstrated, I believe, that the rôle of mathematics in high school physics is somewhat different from that which it assumes in the colleges, and is decidedly different from its dominant position on the frontiers of physics research. By illustration, the attempt was made to bring out the nature of this rôle, i.e., that mathematics is an invaluable tool for simplifying, clarifying, and enriching various aspects of the subject.

A consideration of the amount of mathematics training needed for this function followed, and the conclusion was reached that while the average student in physics has been "exposed" to enough mathematics, for various reasons, he is unable to work many quite simple problems. The fault cannot be laid at the door of any one department of the school, being inextricably tied up with the newness of the subject for the student, and with the smallness of transfer. With the idea of improving, if possible, the mathematical preparation, suggestions for slight changes from and additions to the usual algebra course were given.

If, as the result of this chapter, any clearer understanding of the character and scope of the problem and of the relationship between mathematics and physics is attained, the purpose with which its writing was undertaken will have been realized.

POLYGONAL FORMS

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The empirical aesthetic formula [1]

$$M = O/C$$

where O is the "order" attributed to certain aesthetic factors (such as symmetry, etc.) with proper weights attached, where C is the "complexity," and M is the "aesthetic measure" itself, finds its simplest possible application in the interesting, if elementary, aesthetic question of polygonal form. Our aim in this chapter is to deal with this case, and exhibit the result of the rigid application of the formula to ninety typical polygonal forms, ordered according to decreasing values of M (pp. 190-195). If the reader finds a gradual diminution in attractiveness in passing from the first polygon (No. 1) to the last (No. 90), the formula may be regarded as substantiated.

The judgment of students in two graduate courses, held at Columbia University (summer 1929) and Harvard (summer 1930), seems to indicate the validity of the formula. I wish to express here appreciation of the cordial coöperation which I received from the students in these classes. In one instance, that of the right triangle resting on a side (No. 70), the rating was felt to be too low. If, however, the context supposed in this connection, namely, that a *single* polygon is used as a *tile* in vertical position, is kept in mind, it will be clear that while the right triangle is valued highly as an element in composition, it is scarcely ever used in this particular way, i.e., in isolation.

In judging the validity of the formula it is necessary of course to eliminate all accidental connotations, such as the religious one of the crosses in the list, that of a rectangular box suggested by

[1] *Proceedings of the International Mathematical Congress*, Bologna, 1928. For a discussion of the psychological basis of the formula, see an article "A Mathematical Approach to Aesthetics," shortly to appear in *Scientia*. I expect to publish various applications in book form as soon as possible.—AUTHOR.

No. 37, and the like. Furthermore, the first effect of novelty must be discounted.

1. Preliminary Requirements. If the problem of classification of polygonal forms is to have reasonably precise meaning, some particular representation must be chosen. We shall accordingly imagine that we have before us a collection of blue porcelain tiles, of uniform material and size. It is easy to consider these solely in their aspect of pure form.

A further requirement must be imposed in order to fix the psychological state of the "normal observer." Such a polygonal tile produces a somewhat different impression when it is seen upon a table than when it is seen against a vertical wall. In fact, such a tile lying upon the table would be viewed from various angles, while on the vertical wall it would have a single favorable orientation. Therefore it is desirable to think of the polygonal tiles as situated in vertical position against a wall. In general, the selected orientation will of course be the most favorable, although it need not be so, as for instance in No. 85 of the list.

Perhaps the actual use of the polygon for decorative purposes which most nearly conforms to these conditions is that in which some selected porcelain tile is set at regular intervals along a stuccoed wall.

Just as in all other aesthetic fields, a certain degree of familiarity with the various types of objects involved is required before the aesthetic judgment becomes consistent and certain. The ninety polygons listed in order of decreasing aesthetic measure will furnish a fair idea of the extent and variety of polygonal forms.

It is clear that when these requirements are satisfied the problem of polygonal form becomes a legitimate one.

2. Triangular Form. With these preliminaries disposed of, let us turn to a determination of the principal types of aesthetic factors affecting our enjoyment of polygonal form. Once such a determination has been made, we will be prepared to assess the relative importance of these factors, and to formulate an appropriate aesthetic measure such as we are seeking. We will begin with the simplest class of polygons, namely the triangles.

Now triangles are usually classified as being either isosceles and so having at least two sides equal, or scalene. Clearly the isosceles triangles are more interesting from the point of view of aesthetic form. The proper orientation of an isosceles triangle is naturally one in which two equal sides are inclined at the same angle to the vertical. This is the case for each of the triangles *(a)*, *(b)*, *(c)* of the adjoining figure. In all of these, the triangle "rests"

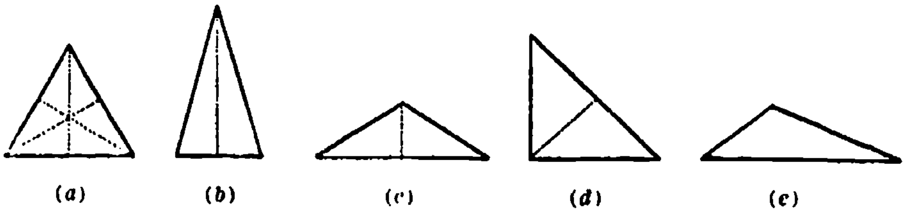


FIGURE 1

upon a horizontal side. However, if these triangles are inverted, the equal sides will again be inclined at the same angle to the vertical. It is readily verified that this second reversed orientation is also satisfactory, and that these two are superior to all others. With these orientations only do we obtain "symmetry about a vertical axis." This is clearly a *desideratum* of first importance.

Obviously if a symmetrical figure be rotated about the axis l of symmetry through 180°, it returns to its initial position.

Judgments of symmetry about a vertical axis are constantly being made in our everyday experience. Let us recall, for example, how quickly we become aware of any slight asymmetry in the human face. Thus the association "symmetry about a vertical axis" is intuitive, and is pleasing to a notable degree in almost every instance.

If the isosceles triangle *(b)* be made to rest upon one of the two equal sides, there still remains the feeling that the triangle is in equilibrium, although the symmetry about the vertical axis is thereby destroyed. It will be observed, furthermore, that the symmetry about the inclined axis is scarcely noted by the eye and is not felt favorably. Thus the triangle in its new orientation makes much the same impression as any scalene triangle which rests upon a horizontal side [compare with *(e)*]. The indifference of the eye to such an inclined axis of symmetry is also evidenced by the isosceles right triangle *(d)* with one of its equal sides horizontal.

On the other hand, if the isosceles triangle *(b)* be given any orientation whatsoever other than the two with vertical symmetry and the third just considered, there is dissatisfaction because of the lack of equilibrium.

For the isosceles triangle (*c*), however, there are only two orientations in which it seems to be in equilibrium, namely, the two with vertical symmetry. In fact, if the triangle (*c*) be made to rest upon one of its two equal sides, the center of area falls too far to left or right as the case may be, and this fact gives rise to an unpleasant lack of equilibrium.

The association "equilibrium" is also one constantly made in everyday experience, and is felt pleasantly.

Among the various shapes of isosceles triangles, the equilateral triangle (*a*) with all sides equal stands out as one possessing peculiar interest. If such an equilateral triangle be set in a position which is not symmetrical about a vertical axis, all the pleasure in its exceptional symmetrical quality disappears. However, once the favorable orientation is taken, this quality is fully enjoyed. In the first place the three axes of symmetry are noted. But it is the "rotational symmetry" which is especially effective.

In the case of the equilateral triangle the center C of rotation is the point of intersection of the three axes of symmetry, and the angle of rotation a is evidently 120° or one-third of a complete revolution.

The rather occult association "rotational symmetry" is often made visually in everyday experience. The appreciation of the form of the circle may perhaps be regarded as fundamentally based upon this association.

Among the isosceles triangles which are not equilateral, there seems to be little to choose in respect to aesthetic merit. It does not appear to be a matter of importance whether the angle between the two equal sides is acute as in (*b*), obtuse as in (*c*), or a right angle.

The scalene triangles are readily disposed of. The best position is one in which the triangle rests upon a horizontal side long enough for the triangle to be in equilibrium. The right triangle with vertical and horizontal side is obviously the best among the scalene triangles [note the triangle (*d*)]. From one point of view this is because an unfavorable factor enters into the general scalene triangle of type (*e*) due to the presence of *three* unrelated directions. The treatment of this negative factor of "diversity of directions" is rather technical since it involves the notion of the "group" of a polygon (Sections 14, 16).

Thus the various types of triangles in a vertical plane can be grouped in the following five given classes in descending order of aesthetic value: (1) an equilateral triangle with vertical axis of symmetry; (2) an isosceles triangle with vertical axis of symmetry; (3) a right triangle with vertical and horizontal side; (4) a triangle which is without vertical axis of symmetry and rests upon a sufficiently long horizontal side to insure the feeling of equilibrium; (5) any triangle which lacks equilibrium. The triangles of the first three classes are definitely pleasing; those of the fourth class are perhaps to be considered indifferent in quality; and those of the fifth class are definitely displeasing. Since it is a natural requirement that the best orientation of any triangle be selected, the fourth class will contain all the scalene triangles without a right angle, and the fifth class will not enter into consideration.

It has been tacitly assumed in the above analysis of triangular form that no side of the triangle is extremely small in comparison to the two other sides, and that no angle is very small or very near to 180°. These are obvious prerequisites if the triangle is to be characteristic. If they are not met, the triangle approximates in form to a straight line and the effect is definitely disagreeable, because of ambiguity.

We are now in a position to list the aesthetic factors that have been thus far encountered: vertical symmetry (+), inclined symmetry (0), equilibrium (+), rotational symmetry (+), perpendicular sides (0), diversity of directions (—), small sides (—), small angles or angles nearly 180° (—).

Here and later we use the symbol (+) to indicate that the corresponding association or element of order operates to increase aesthetic value, the symbol (0) to indicate that it is without substantial effect, and the symbol (—) to indicate that it diminishes aesthetic value. Such associations or elements of order will accordingly have a positive index in O if the symbol is (+), and a negative index if the symbol is (—).

3. Plato's Favorite Triangle. It cannot be emphasized too much that the classification of the various forms of triangle given above takes into account only the simplest and most natural aesthetic factors. How completely such a scheme of classification can be upset by the introduction of factors based upon fortuitous associations, is easily illustrated.

Plato in the *Timaeus* says: "Now, the one which we maintain to be the most beautiful of all the many triangles (and we need not speak of others) is that of which the double forms a third triangle which is equilateral." The context makes perfectly clear in what sense this statement is to be interpreted: If one judges the beauty of a triangle by its power to furnish other interesting geometrical figures by combination, there is no other triangle comparable with this favorite triangle of Plato. For out of it can be built (Figure 2) the equilateral triangle, the rectangle, the paral-

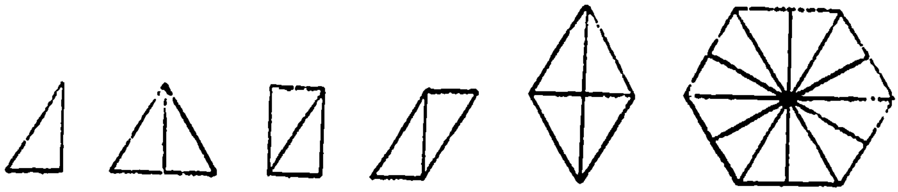


FIGURE 2

lelogram, the diamond, and the regular hexagon among polygons, as well as three of the five regular solids. This power in combination was peculiarly significant to Plato, who valued it for purposes of cosmological speculation. It was on such a mystical view that he based his aesthetic preference.

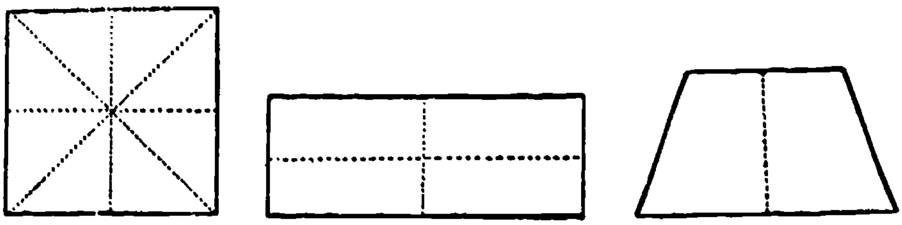
Yet it may well be doubted whether persons not having his particular philosophic outlook would agree with Plato. In fact, it appears that this scalene right triangle is not superior to the general right triangle for the aesthetic problem under consideration.

4. The Scalene Triangle in Japanese Art. It is well known that the Japanese prefer to use asymmetric form rather than the too purely symmetric. Indeed, in all art, whether Eastern or Western, obvious symmetry tends to become tiresome.

In particular it has been said that all Japanese composition is based upon the scalene triangle. Is this fact in agreement with the classification effected above which concedes aesthetic superiority to the isosceles and in particular to the equilateral triangle? The answer seems to be plain: When used as an element of composition in painting the isosceles triangle may introduce an adventitious element of symmetry which is disturbing to the general *motif*. But, in the much more elementary question of triangular form *per se*, the general opinion, at least in the West, is in favor of the equilateral and isosceles triangle rather than the scalene triangle.

Recently while in Japan I was fortunate enough to be able to question one of the greatest Japanese painters, Takenouchi Seiho, in this matter, and I understood from him that the same opinion would doubtless be held in Japan.

5. The Form of Quadrilaterals. Let us turn next to the consideration of the form of quadrilaterals, and let us examine first those types in which there is symmetry about a vertical axis. There are two cases. In the first case at least one side of the quadrilateral intersects the axis of symmetry. Evidently such a side must be perpendicular to the axis of symmetry. Furthermore, there must then be a second opposite side which is also perpendicular to the axis. Thus the general possibility is that of a symmetric trapezoid given by *(c)* of Figure 3. This trapezoid may,



(*a*) (*b*) (*c*)

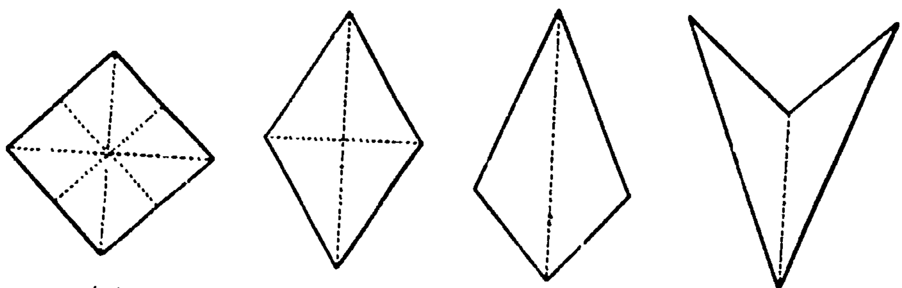
FIGURE 3

however, take the form of a rectangle or square illustrated by (*b*) and (*a*) respectively. It is to be observed that the rectangle possesses a further horizontal axis of symmetry, while the square possesses not only a horizontal and vertical axis of symmetry but also two axes inclined at 45° to the horizontal direction. Likewise, both rectangle and square have rotational symmetry, the angles of rotation being 180° and 90° respectively. Judging by the extent of symmetry involved, we should expect to find the square to be the best in form, the rectangle excellent, and both superior in aesthetic quality to the symmetrical trapezoid. Such a relative rating coincides with my own aesthetic judgment and that of many others.

Both the rectangle and the square have central symmetry. Any figure with rotational symmetry whose least angle of rotation is contained an even number of times in 360° will possess central symmetry; cases in point are the rectangle and the square, since 180° and 90° are contained respectively 4 and 2 times in 360°. On the other hand, any figure with rotational symmetry whose least angle of rotation is contained an odd number of times in 360° will not possess central symmetry; for example, the equilateral triangle is a case in point, with a least angle of rotation 120° which is contained 3 times in 360°.

It has been claimed that the rectangle is a form superior to the square and even that certain rectangles such as the "Golden Rectangle" excel all others. I shall indicate later in what sense, if any, such statements can be valid (Section 6).

In the second case of symmetry about the vertical axis, the quadrilateral has two of its vertices on the axis of symmetry, but none of the sides intersects the axis. Here the general possibility is indicated by (*c*) and (*d*) of the figure below in which quadri-



(*a*) (*b*) (*c*) (*d*)

FIGURE 4

lateral (*c*) is convex and (*d*) is reëntrant. The first of these may, however, reduce to the equilateral quadrilateral or diamond as in (*b*), or even to the square (*a*) with sides inclined at 45° to the horizontal direction.

Of the two general cases represented by the quadrilaterals (*c*) and (*d*) in Figure 4, it is clear that the convex type (*c*) is definitely superior to the alternative reëntrant type of quadrilateral (*d*). This reëntrant character evidently operates so that the quadrilateral suggests a triangle from which a triangular "niche" has been removed. In general each polygon lying between a given reëntrant polygon and the minimum convex polygon which encloses it is termed a "niche" of the reëntrant polygon. A rubber band stretched around the given polygon will take the form of the minimum enclosing convex polygon.

It is not the mere fact that the quadrilateral is reëntrant which is decisively unfavorable. Consider, for example, the hexagon or six-pointed star (see No. 6, page 190). This star is evidently highly pleasing in form and yet it is reëntrant. It will be noted, however, that every side of the star although of "reëntrant" type is "supported" by another side which lies in the same straight line, while this is not true of the reëntrant sides of the quadrilateral.

In the comparison of quadrilaterals of types (*a*) to (*d*) on page 172 we find, as we should expect, that the square (*a*) and the diamond (*b*) in the orientations indicated are markedly superior to the quadrilaterals (*c*) and (*d*) already discussed.

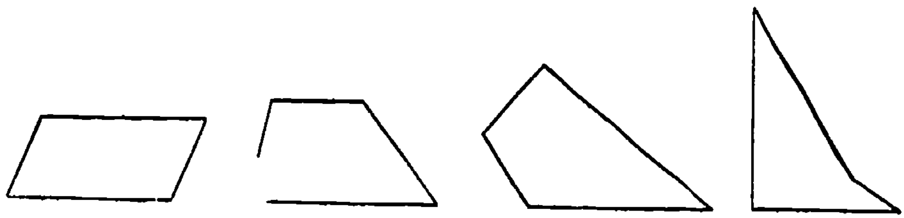
However, it seems to me to be difficult to say whether or not the square so situated is better in form than the diamond, despite the fact that on the score of symmetry alone the square holds higher rank. As far as I can analyze my own impressions, I am led to the following explanation of this uncertainty in aesthetic judgment: For me and many others the orientation of the square with sides vertical and horizontal is superior to the orientation in which its sides are inclined at an angle of 45° to the horizontal direction; indeed, this superiority will appear in consequence of the definition of aesthetic measure adopted later.[2] Hence a certain feeling of displeasure is experienced in the contemplation of the square in the orientation (*a*) above, just because it is found in an inferior orientation and because it would be *so easy* to alter it for the better. If one could abstract one's feeling completely from this association, which is really irrelevant, I believe the square (*a*) would actually impress one as being superior in aesthetic quality to the diamond (*b*).

There remain for discussion those quadrilaterals which have no vertical axis of symmetry. Here, as in the case of the scalene triangle, attention can be limited to cases in which the quadrilateral rests on a sufficiently long horizontal side so that it appears to be in stable equilibrium, for otherwise the quadrilaterals are definitely displeasing.

It is readily found that the only quadrilaterals of this sort which possess rotational symmetry are the parallelograms, which are illustrated by (*a*) in Figure 5. Evidently the characteristic angle of rotation for a parallelogram is 180°. The parallelogram is the simplest type of polygon possessing rotational symmetry but not symmetry about any axis, and it evidently stands first among the types (*a*) to (*d*). Next to the parallelogram in aesthetic quality

[2] See pages 186-187.

follows the trapezoid, with two sides parallel as in (*b*); evidently the association "parallel" is one made constantly and intuitively in everyday experience. Then follow the general convex quadrilateral like (*c*), and finally the reëntrant case illustrated by (*d*).



(*a*) (*b*) (*c*) (*d*)

FIGURE 5

It will be observed that the presence of an isolated right angle or of two equal sides, as in (*c*), is without special influence.

We have now examined the various types of quadrilaterals and arranged those of each type in order of preference. It remains to compare briefly those of different types. On the basis of my own aesthetic judgment I am led to arrange them in the following order of diminishing aesthetic value: the square; the rectangle; the diamond; the convex quadrilateral symmetric about an axis through two opposite vertices, the symmetric trapezoid and the parallelogram; the reëntrant quadrilateral symmetric about an axis; the convex quadrilateral without symmetry; the reëntrant quadrilateral without symmetry. Of these, the last two types are definitely unsatisfactory. Here I assume that the quadrilaterals are placed in the most favorable position, of course. This relative arrangement is that assigned by the aesthetic formula.

We shall not attempt at this stage to compare triangles and quadrilaterals with one another.

There are two new types of elements of order brought to light by our examination of quadrilaterals. The first is of negative type (−) and operates when the quadrilateral is reëntrant. Further insight into the nature of this element will be obtained in a following section. The second is connected with the parallelism of sides and is of positive type (+). It is more convenient, however, to regard this second element on its negative side, when it is aptly characterized as "diversity of directions of sides" ( ).[3] Evidently,

[3] See page 188

other things being equal, the more parallelisms of sides there are, the less will be the diversity of direction of the sides.

It has furthermore appeared that the mere equality of sides is an indifferent factor (0) for the quadrilaterals and also for polygons having more than four sides. This stands in apparently sharp distinction from the case of the triangle. The reason for the difference is to be found in the fact that only for the triangle does equality of two sides insure symmetry.

6. The Golden Rectangle. It will perhaps have been noted that thus far in our analysis the precise dimensions of polygons have scarcely been considered. The question whether or not these dimensions have aesthetic significance has frequently been asked, particularly in reference to the rectangle.

The German psychologist Fechner conducted elaborate experiments to ascertain if possible the most satisfactory rectangular shape, inclusive of the square. His results may be briefly summarized as follows: The square and more especially a rectangle of dimensions having the ratio of longer to shorter side of about 8 to 5 were generally adjudged to be the best among the various rectangular shapes.

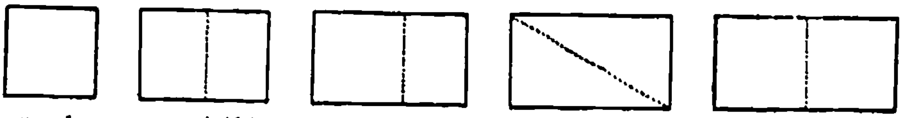
Now the so-called Golden Rectangle is characterized by the ratio 1.618 . . . which is nearly 8 to 5. Is it not perhaps true that there resides in it some special occult beauty which makes it superior to all others?

The interesting geometrical property that if a square on the shorter side be taken away, another Golden Rectangle remains (see Figure 6) has seemed to some persons to justify the conclusion that it is the most beautiful rectangle. This view seems to me without adequate basis.

In order to understand the matter, it is well to keep in mind the other special rectangle favored by Plato, made of the two halves of an equilateral triangle. For it, the ratio is 1.732 . . . . It is also well to keep in mind the rectangle with ratio 1.414 . . . , which, if divided in two equal rectangles by a line parallel to the two shorter sides gives two rectangles of the same shape as the original rectangle. Here the ratio is nearly 7 to 5. Furthermore, the rectangle made up of two squares, with the ratio 2 to 1, is also to be noted.

Thus we find five rectangles (we include the square) with simple geometric properties. These are represented in Figure 6, in which the ratio r of the longer side to the shorter is indicated in each case.

It can perhaps be justly said that a shape not suggesting a simple numerical ratio like 1 to 1 or 2 to 1 is desirable in many connections where the rectangle is used as an element in composition. In that event the square and the double of a square are



$r = 1$ $r = 1.414$ $r = 1.618$ $r = 1.732$ $r = 2$

FIGURE 6

excluded. Moreover, perhaps the rectangle approved by Plato might not serve since it does not differ sufficiently from the double of a square. But this still leaves a wide range of choice, embracing the cases $r = 1.414$ and also $r = 1.618$, for instance. Of these two I slightly prefer the rectangle with ratio 1.414 to the Golden Rectangle with ratio 1.618.

Now it may be that certain persons, through their acquaintance with and liking for Greek art, have come to individualize and identify with fair approximation the particular shape embodied in the Golden Rectangle. For such persons an intuitive association of purely accidental character would be established in favor of the Golden Rectangle. Only by assuming that a number of his experimental subjects were of this type can I understand the experimental results of Fechner in favor of a particular ratio of nearly 8 to 5.

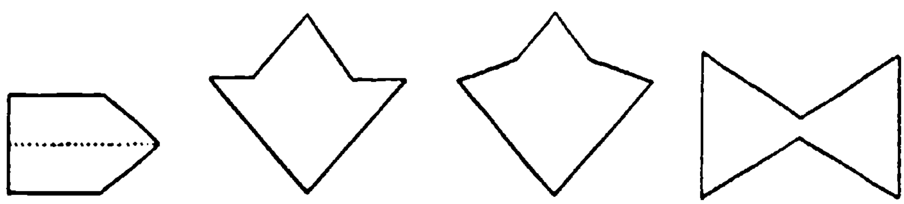
In comparing the square and the rectangle, it should not be forgotten that the rectangles contain an infinitude of shapes, dependent on the ratio of the sides, whereas the square presents but a single shape. Hence the rectangles provide a much more flexible instrument in design than the square does. It is, for example, obvious that when a rectangular frame is used for a portrait the square shape is in general less suitable than that of a rectangle with the height greater than the breadth. However, I believe that the square is much more often used than any other single rectangular form, such as the Golden Rectangle.

These remarks do not quite do full justice to the special forms of the rectangles when these are not used singly but in combination with other polygonal forms. For instance, the arrangement of two adjoining rectangular windows with $r = 1.414$ so as to form a single such window with $r = 1.414$ might be decidedly pleasing in certain architectural effects, because the same shape is discovered in a new aspect. Similarly, three adjoining windows, the central one being square and the outer ones equal Golden Rectangles with short sides horizontal, might prove very pleasing for a like reason.

7. The Form of Five- and Six-sided Polygons. Our survey of triangles and quadrilaterals has brought to light a number of the essential aesthetic factors which operate in the case of more complicated polygons. So far, these have all been in the nature of associations or elements of order. It would be tedious to continue with our analysis, step by step. If we did so, the facts for five- and six-sided polygons would be found to be on the whole similar to those already noted. We shall be content, therefore, to note those features which are not illustrated by the simpler polygons of three or four sides, and then to consider the other features which are illustrated only by still more complicated polygons.

To begin with, let us recall our previous conclusion that symmetry about an inclined axis has little or no significance in itself. It is true that when there is symmetry about the vertical axis also, the matter is not so clear; but in that event, there will also be rotational symmetry, so that such symmetry about an inclined axis can be considered always as arising out of a combination of symmetry about the vertical and rotational symmetry, in case it operates effectively.

Thus the question arises immediately as to the significance of symmetry about a horizontal axis when there is no vertical sym-



(a) (b) (c) (d)

FIGURE 7

metry. This case is illustrated most simply by the pentagonal polygon (a) in Figure 7. In the first place, it is clear that the symmetry about the horizontal axis is much more easily appraised by the eye than the symmetry about an axis in any other direction except the vertical. Notwithstanding this fact, the symmetry about the horizontal axis is regarded indifferently or with antipathy, because it is felt that the polygon is not properly placed. Hence we are led, as before, to rate symmetry about a horizontal axis as in itself an aesthetic factor of indifferent type (0).

A second factor already briefly alluded to[4] is effectively isolated by a comparison of the two reëntrant hexagonal polygons (*b*) and (*c*). These are both of the same general type, but only in the first case (*b*) do two of the four reëntrant sides lie in a straight line and so support one another. It is obvious that (*b*) is notably superior to (*c*) just on this account.

In general, then, we may expect reëntrant sides which are not supported by other sides in the same straight line (Section 5) to operate unfavorably and so to correspond to a factor of negative type (—).

The psychological explanation of this situation appears to be evident. In general the association "reëntrant" is not a pleasant one. But if, when the eye follows a reëntrant side, another side is discovered in the same straight line, there is a compensating feeling of satisfaction. Thus only the unsupported reëntrant sides need to be considered as producing a definitely negative tone of feeling.

The third factor is illustrated by (*d*) of the same figure, which is essentially made up of two triangles only overlapping slightly near a vertex of each. In consequence, the six-sided polygon is not characteristic and this fact is felt unfavorably. It will be noted that the two vertices are then very near to one another and also to near-by sides. Hence there is a factor dependent on too great nearness of vertices to other vertices or sides, which is of negative type (—). We have already observed the special instance when one of the sides of a triangle is excessively short.

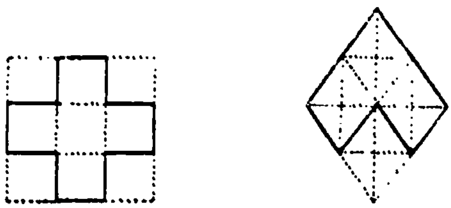
8. More Complicated Forms. The ninety polygons listed in descending order of aesthetic measure according to the empirical rules later adopted, present graphically the principal types of polygons, and may be regarded as reasonably complete for polygons of not more than nine sides. The survey of these polygons brings to light a few further aesthetic factors of importance.

One of the new items is the obviously increasing complexity itself, which, beyond a certain point, is found to be burdensome.

[4] See page 172.

In this connection we may note the fact that the convex polygon of many sides is more likely to be pleasing than the reëntrant one, particularly if the latter contains a diversity of niches.

A second very important new factor makes its appearance when the polygon is directly related to some uniform network of horizontal and vertical lines, as in the case of the Greek cross [see (*a*), Figure 8], or else is closely related to a uniform diamond network



(*a*) (*b*)

FIGURE 8

[see (*b*)] with its sides equally inclined to the vertical, this diamond network in turn suggesting a uniform horizontal-vertical network.

Evidently the aesthetic factor of close relationship to a uniform horizontal-vertical or diamond network plays a fundamental part and enhances the aesthetic value of many polygons listed, e.g., the square (No. 1), the rectangle (No. 2), the diamond (No. 4), the hexagram (No. 6), the Greek cross (No. 9), the swastika emblem (No. 41), etc. In the case of the square, rectangle, and diamond, the validity of this association is perhaps debatable. But, so frequently do we see these polygons used in conjunction with a network that it seems proper to regard them as suggesting relationship to a uniform horizontal-vertical or diamond network. On the other hand, it is evidently not legitimate to regard the association with a uniform diamond network as possessing interest equal to that of association with a horizontal-vertical network. It is with these facts in mind that the empirical rule dealing with the aesthetic factor of relationship to such a network will be formulated. The associational basis of this factor in everyday experience is obvious. Systems of lines placed in the regular array of a network are constantly met with, and their relationship to one another is intuitively appreciated.

In the consideration of such more complicated polygons it appears a.. that some kind of symmetry is always required if the polygonal form is to be at all attractive. When this requirement is not met, no degree of relationship to a horizontal-vertical network, for instance, can entirely offset the deficiency. Thus, if symmetry is lacking, the fact appears as a definite negative aesthetic factor which must be taken account of.

Evidently a further actual aesthetic factor in many cases is some accidental association, such as is present, for example, in the case of the cross and the swastika. The mathematical theory takes no account of such completely indefinable "elements of order," although they have a definite aesthetic effect.

9. On the Structure of the Aesthetic Formula. According to the general theory proposed in the first chapter, we seek an aesthetic formula of the type $M = O/C$ where M is the aesthetic measure, O is the order, and C is the complexity. In the case of polygonal form before us, O will be separated into five elements,

$$O = V + E + R + HV - F$$

The aesthetic factors encountered above are correlated in the following way with C and these five elements:

- C: complexity.
- V: vertical symmetry (+).
- E: equilibrium (+).
- R: rotational symmetry (+).
- HV: close relation to a horizontal-vertical network (+).
- F: unsatisfactory form involving some of the following factors: too small distances from vertices to vertices or sides (−), or angles too near 0° or 180°, or any other ambiguity of form; diversity of niches (−); unsupported reëntrant sides (−); diversity of directions (−); lack of symmetry (−).

It will be observed that the term F involving the general attributes of unsatisfactory form is an "omnium gatherum" for all the negative aesthetic factors which have been noted.

The various indifferent factors of type (0) play no part of course. Some of these are equality of sides, perpendicularity of sides, and inclined or horizontal symmetry (without vertical axis of symmetry).

In the course of the technical evaluation of C, V, E, R, HV, F, and so of M, to which we now proceed, a simple mathematical concept, namely, that of the group of motions of the given polygon, will be introduced. This concept forms a basic mathematical adjunct, necessary for the comprehension of the problem before us.

The aesthetic measure M defined by the formu'a will turn out to depend upon the orientation of the polygon. In case an axis of symmetry exists, the highest aesthetic measure of a particular polygon will invariably be found when such an axis is taken in the vertical direction. But the same measure will be obtained if this orientation is reversed. This seems to be in accordance with the facts which are observed. An apparent exception is furnished by the Roman cross (No. 49) and certain other polygons, when an inversion appears to diminish notably the aesthetic value. But this will be found to be due to the fact that an important connotative element of order is thereby removed; for instance, in the case of the cross it is the religious association. Our conclusion, therefore, is that these exceptions are apparent rather than real.

10. The Complexity C. The complexity C of a polygon will be defined as the least number of indefinitely extended straight lines which contain all the sides of the polygon. Thus for any quadrilateral the complexity is evidently 4; for the Greek cross (No. 9) the complexity is 8, although the number of sides in the ordinary sense is 12; for the pinwheel-shaped figure (No. 53), the complexity is evidently 10; and so on.

The psychological reasonableness of this empirical rule is evident. For convex polygons, and also for polygons which are not convex but which do not possess any two sides that are situated in the same straight line, the complexity C is merely the number of sides. As the eye follows the contour of the polygon in looking at the various sides in succession, the effort involved would appear to be proportional to the number of sides. On the other hand, if there are two or more sides on one and the same straight line, the eye follows these in one motion. For example, in the case of the Greek or Roman cross, the eye might regard it as made up of two rectangles. These considerations suggest that the definition chosen for the complexity C is appropriate.

11. The Element V of Vertical Symmetry. The organization of the entire polygon which results from vertical symmetry is obvious to the eye. By long practice we have become accustomed to appreciating symmetry of this sort immediately. On this account the element V is particularly significant.

We shall give to V the value 1 if the polygon possesses symmetry about the vertical axis, and the value 0 in the contrary case. In other words, the element V will be a unit element of order, and, since there exist various polygons of pleasing quality, such as the swastika, which do not have vertical symmetry, we shall assign the value 0 rather than some negative value to V when there is no such symmetry.

A large proportion of the polygons listed possess such vertical symmetry, and it can be verified that the presence of such symmetry is immediately and favorably recognized.

12. The Element E of Equilibrium. We consider next the second element E of order concerned with equilibrium. It has been previously observed that when the polygon has vertical symmetry or when it rests upon a sufficiently extended horizontal base, it is felt to be in equilibrium.

In order to specify completely the requirements for equilibrium, we note first that it is *optical* equilibrium which is referred to, rather than ordinary *mechanical* equilibrium. For example, the pinwheel polygon No. 53 would actually be in (unstable) mechanical equilibrium if turned through an angle of 45°, inasmuch as the center of area would lie directly above the lowest point. Nevertheless, it does not give the optical impression of equilibrium, since the eye is not accustomed to estimating accurately the balance of figures acted upon by gravity.

What is really needed in order that the feeling of complete equilibrium be induced, is either that there be symmetry about the vertical, or that the extreme points of support at the bottom of the polygon are sufficiently far removed from one another, with the center of area lying well between the vertical lines through these two extreme points.

In order to state sufficient requirements for this, we shall agree that complete optical equilibrium will be induced if the center of area lies not only between these two lines, but at a distance from either of them, at least one-sixth that of the total horizontal breadth of the polygon. If this somewhat arbitrary condition is satisfied, as well as in the case of vertical symmetry, we shall give E the value $+1$. If the polygon does not satisfy these conditions but is in equilibrium in the ordinary mechanical sense, we shall take E to be 0. Otherwise we shall take E to be -1, inasmuch as the lack of equilibrium is then definitely objectionable.

In general the best orientation of a polygon will be one of complete equilibrium. Among the ninety polygons listed, this is the case without exception.

13. The Element *R* of Rotational Symmetry. In the case when the polygon possesses rotational symmetry, there is evidently a least angle of rotation $360°/q$ which is an exact divisor of 360°. If the corresponding rotation be repeated successively q times, the polygon will not only be returned as a whole to its initial position, but a complete revolution will have been effected, so that every point goes back to where it was at the outset.

There is a certain significant similarity between the rotational symmetry which we are now considering and the axial symmetry already referred to. In order to make this plain let us recall that if a polygon be rotated through 180° about an axis of symmetry, it also will return to its initial position. Thus the test for both kinds of symmetry is that a certain specified motion of rotation restores the polygon to its initial position. In the case of rotational symmetry the axis of rotation is perpendicular to the plane of the polygon at its center of area and the rotation is in the plane of the polygon; while in the case of axial symmetry the axis of rotation is an axis of symmetry in the plane, and the rotation is a half revolution.

A consideration of the various possibilities when a polygon possesses symmetry of rotation shows that there are several cases which must be distinguished from one another in their aesthetic effect.

The first and simplest case is that in which the polygon has not only symmetry of rotation but symmetry about an axis (vertical) as well. In this case, illustrated by many polygons of the list, such a symmetry of rotation is immediately and favorably felt by the observer. Evidently an axis of symmetry is moved into a new axis of symmetry when the polygon is carried into itself by an allowed rotation in its plane.

A more detailed mathematical consideration of this case which the reader can easily verify by means of a few examples shows that in all cases there are precisely q distinct axes of symmetry with an angle $360°/2q$ between successive axes. If q is odd, all of these axes are obtained from a single one by means of rotations of the polygon into itself in its plane; whereas, when q is even there are two sets of $q/2$ axes of symmetry at angles $360°/q$ apart, the axes of one set bisecting the angles between those of the other. The rectangle illustrates the general situation for $q = 2$, while the regular polygon of q sides illustrates it for $q > 2$.

There is also a second case when the rotational element R is felt almost equally favorably, despite the lack of true vertical symmetry; namely, when the minimum convex polygon which incloses the given polygon does not abut upon any of its niches, and is symmetric about a vertical axis. Polygons Nos. 41, 50, 51, and 69 illustrate this case. Here the inclosing convex polygon is so strongly suggested, with its q axes of symmetry, as to suggest vividly the rotational element.

On the other hand, even though the minimum inclosing polygon is symmetric about a vertical axis, the same effect is not felt if the niches abut on its vertices. This third case is illustrated by polygons Nos. 53, 67, 85, 88, 90. In partial explanation of this difference in effect it may be observed that in the first two cases the center of the polygon is clearly defined, either by means of the axes of symmetry of the given polygon or by means of those of the minimum enclosing polygon which are strongly suggested. This circumstance operates advantageously.

In both of the first two cases we shall take the element R as $q/2$ so long as q does not exceed 6. For q equal to 6 or greater, we take $R = 3$, since the effect of the rotational element is limited and seems to attain its maximum when q is 6. The mathematical reason for the particular choice of $q/2$ as the value of R when q does not exceed 6 is alluded to in the next section.

We pass next to the further consideration of those pelygons falling under the last case. Two possibilities need to be distinguished according as q is even or odd respectively, thus giving rise to a third and fourth case respectively.

When q is even, there is central symmetry, and such central symmetry is appreciated immediately. In particular, it enables the observer to fix the center accurately. Here there are no axes of symmetry, actual or suggested. Since it is only these that bring out clearly the rotational symmetry, the rotational symmetry as such plays only a small rôle; and so we take $R = 1$ in the third case, whatever be the value of q so long as it is even. The justification from the mathematical point of view of the choice $R = 1$ in this case will not be attempted here.

Polygons Nos. 39, 45, 48, 53, 65, 67, 74, 77, 84, 85 illustrate this third case $R = 1$.

On the other hand, the four polygons, Nos. 79, 88, 89, and 90, illustrate the convex and reëntrant types in the fourth case. It will be observed that even in the convex type one is scarcely aware of the rotational symmetry.

Thus in this fourth case, as well as in any case where there is no rotational symmetry whatever, we are led to take $R = 0$.

14. The Group of Motions of a Polygon. It has been seen that either a rotation of a polygon about one of its axes of symmetry through 180°, or a rotation of the polygon about its center of area through a multiple of the fundamental angle of rotation $360°/q$, will return the polygon to its original position. There are no other motions which can leave a polygon unaltered. Among these motions we will list (by convention) the particular motion I which moves no point, as well as the other motions A, B, . . . , if such there be.

Definition of the group of motions of a polygon. The collection of rotations of a polygon about each of its axes of symmetry through 180°, and of the rotations about its center of area through any angle of rotation, will be termed the "group of motions" of the polygon.

The group of motions of a polygon has the fundamental property that two such motions A and B performed successively will also return the polygon to its initial position, and so be equivalent to a single motion C of the group; or symbolically, $AB = C$.

This leads us to two further related definitions:

Definition of conjugate figures. If F be any figure in the plane of a given polygon, which takes the successive positions F, F_A, F_B, . . . under the corresponding motions I, A, B, . . . of the group, the figures F, F_A, F_B, . . . are said to be "conjugate."

Definition of fundamental region. A region of the plane which with its conjugate regions fills the entire plane in which the given polygon lies, but in such a way that these regions do not overlap, is said to be a "fundamental region" for the given group of motions.

It is worth while to give an illustration. For the rectangle with vertical axis of symmetry, the motions of the group (besides I) are evidently the rotations S_V and S_H of 180° about the vertical and horizontal axes of symmetry respectively, and the rotation R through 180° in its plane about the center.

It is evident that two opposite sides of the rectangle are conjugate under this group of motions. Similarly, the four vertices of the rectangle are conjugate under the group. Finally, it is clear that any one of the four quadrants into which the plane is divided by the two axes of symmetry is a fundamental region for this group of motions.

It is evident that these concepts are really necessary for the understanding of the element R above treated. In particular, the quantity $q/2 = R$ is 1 for $q = 2$ when the fundamental region is a half plane, and is inversely proportional to the angular size of the fundamental region. This evaluation of R agrees with the choice of V as 1, because of the fundamental half plane in the case of vertical symmetry.

15. The Element *HV* of Relation to a Network. In many polygons of the list, as has been previously noted, there is obviously a close relationship of the given polygon to a uniform horizontal-vertical network, and this relation is at once recognized as pleasing.

Evidently the corresponding element HV in O is connected with certain motions of the plane in much the same way as the element V is connected with a motion of rotation about a vertical axis, and the element R with a motion of rotation about a center. In fact, such a uniform network returns to its initial position when any one of a variety of translatory motions of the plane is made, while these same motions will take the polygon to a new position in which it may in large measure have the same bounding lines as it had in its first position. This happens when most of the sides of the polygon coincide with the lines of such a network. In an incomplete way, then, the element HV is connected with motions of the plane just as are the elements V and R.

The most favorable case is evidently that in which the polygon has all its sides upon a uniform network of horizontal and vertical lines in such wise that these lines completely fill out a rectangular position of the network. In this case only do we take $HV = 2$. Polygons Nos. 1, 2, 9, 23, 25, 26, 29, etc., illustrate this possibility.

The choice of a value $HV = 2$ seems natural since there are essentially *two* kinds of independent translatory motions which return the network to its original position; namely, a translation to the right or left, and a translation up or down. Any other translation may be regarded as derivable by combination from these two.

A similar case is that in which the sides of the polygon all lie upon the lines of a uniform network formed by two sets of parallel lines equally inclined to the vertical, and fill out a diamond-shaped portion of the network. But the effect here is less favorable so that we take $HV = 1$.

It becomes necessary at this stage to face a somewhat difficult and vexing question, and assign an index in all cases when the polygon is agreeably related to a uniform horizontal-vertical or diamond network. We shall fix upon the following empirical rules. We define HV to be 1 if the polygon fills out a rectangular portion of a horizontal-vertical network in such a way that the first set of conditions holds, with the following possible exceptions: one side of the polygon and its conjugates may fall along diagonals of the rectangular portion or of adjoining rectangles of the network; one vertical and one horizontal line, as well as their conjugates, may not be occupied by a side of the polygon.

Illustrations of this case $HV = 1$ are furnished by polygons Nos. 13, 24, 42, 43, 49, 50, 62, 66 of the list given above.

HV will also be defined to be 1 if the polygon fills out a diamond-shaped portion of a diamond network with the following possible exceptions: one side of the polygon and its conjugates may fall along diagonals of the diamond-shaped portion or of adjoining diamonds of the network; one line in the diamond-shaped portion and its conjugates may not be occupied by a side.

Polygons Nos. 5, 6, 24, 53, 65, 68 are illustrations of this case $HV = 1$.

In every case when HV is 1 we shall demand that at least two lines of either set of the network shall be occupied by a side. In all other cases whatsoever we shall take $HV = 0$.

It is obvious that the above determination of indices for the element HV is largely arbitrary. Nevertheless, it seems to agree with the facts observed.

16. The Element F of General Form. There remains to be treated the factor dealing with general form which we have described as an "omnium gatherum" of unfavorable elements (Section 9).

The case where F is 0 corresponds to satisfactory form. Here the analysis made in the earlier sections suggests the following conditions: the minimum distance from any vertex to any other vertex or side must not be too small—for definiteness, we say it is not to be less than $^1/_{10}$ the maximum distance between points of the polygon; no angle is to be too small or too large—for definiteness we say not less than 20° nor greater than 160°; there is to be at most one niche and its conjugates; there is no unsupported reëntrant side; there are at most two directions and their conjugate directions, provided that horizontal and vertical directions are counted together as one; there is sufficient symmetry to the extent that not both V and R are 0.

It is possible to justify this choice of conditions by reference to the ninety listed polygonal forms. Here we may omit the first two conditions from consideration, since they merely eliminate ambiguity of form. The third condition, which admits only one type of niche, has its origin in the observed fact that only a single niche may occur, as in the Greek cross (No. 9), without the form being thereby impaired. The fourth condition is justified by the fact already noted that any unsupported reëntrant side produces a disagreeable effect. Similarly, if there are three or more distinct types of direction, the polygon appears unsatisfactory, unless two of these are vertical and horizontal, as in No. 43, for instance; this fact gives rise to the fifth condition. Finally, the sixth and last of these conditions is obviously to be based upon the fact already noted, that a lack of effective symmetry is not possible with satisfactory form.

By this rule we take account of all the factors of F as these were listed in Section 9.

We consider next the case in which one and only one of the six conditions fails and that to the least possible extent there may be one type of vertex (and conjugates) for which the first condition fails; or one type of angle for which the second condition fails; or two types of niches; or one unsupported reëntrant side; or three types of direction when vertical and horizontal directions are counted as the same; or both V and R may be 0. Under these circumstances we take F to be 1.

In every other case we take F to be 2.

All the polygons Nos. 1 to 22 inclusive have $F = 0$. No. 24 is the first polygon for which F is 1, because of its having two types of niches; No. 23 is the first which has one type of unsupported reëntrant side. The first case for which F is 1 because of diversity of directions is No. 60. The first for which F is 2 is No. 55 which is also the first for which V and R are 0.

17. Summary. The various elements in the complete formula

$$M = \frac{V + E + R + HV - F}{C}$$

have now been defined. The complete definition might be assembled in brief form. Let us, however, merely recapitulate the main facts:

$V = 1$ if there is vertical symmetry (otherwise 0).

$E = 1$ if there is vertical symmetry or the polygon "rests" on a long enough horizontal side; $E = 0$ if there is "unstable" equilibrium; $E = -1$ if the polygon appears about to fall to one side or the other.

$R = q/2$ if there is rotational symmetry with angle $360°/q$ of rotation, and there are axes of symmetry or axes are strongly suggested (e.g., the swastika, No. 41); except that R is taken as 3 for $q > 6$; in any other case $R = 1$ if there is central symmetry, and $R = 0$ if there is not.

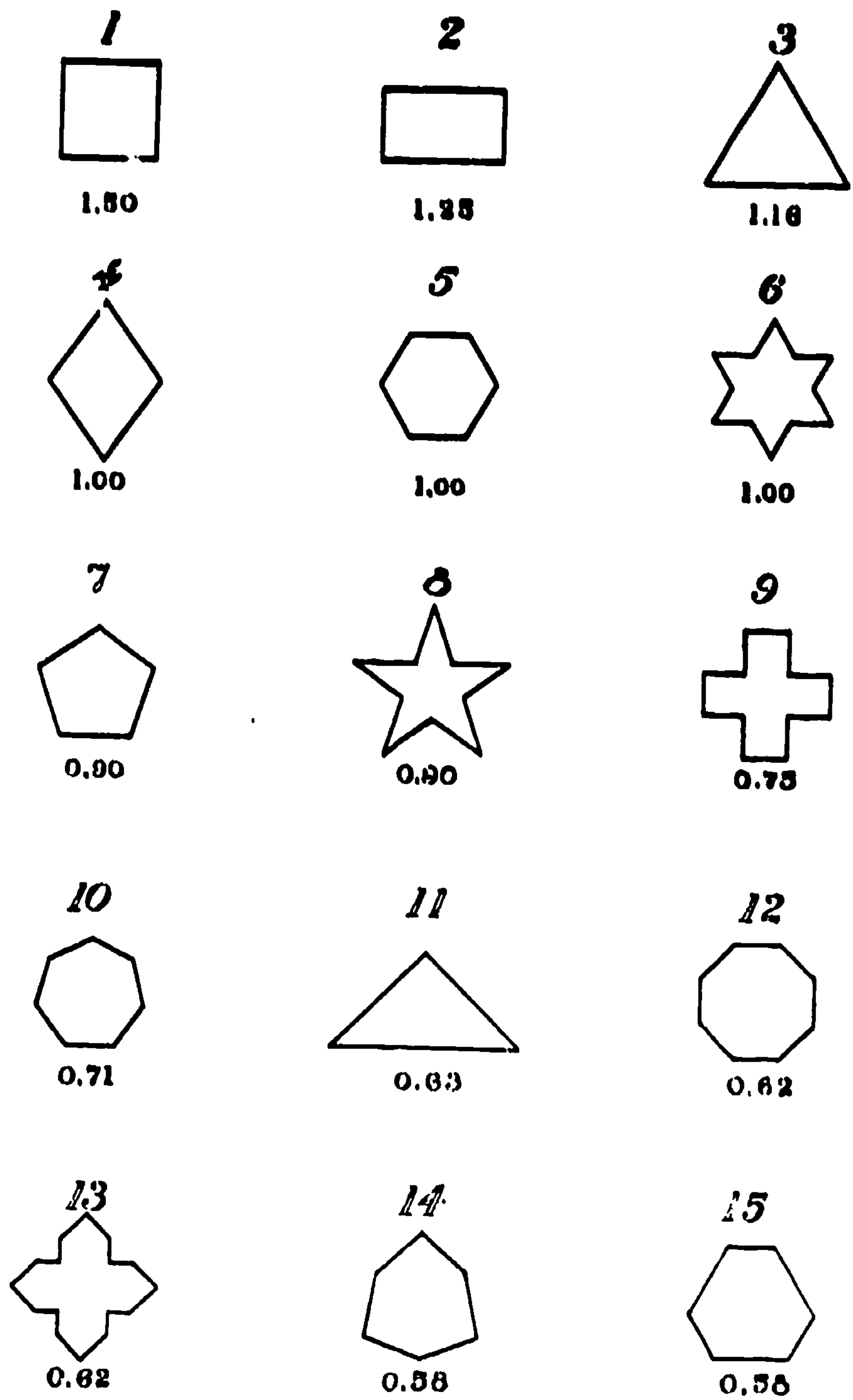
$HV = 2$ if there is complete coincidence of the polygon with a rectangular part of a uniform horizontal-vertical network (e.g., No. 41); $HV = 1$ if there is nearly complete relation to such a network, or complete or nearly complete relation to a uniform diamond network with sides equally inclined to the vertical: HV is 0 otherwise.

$F = 0$ if the form is satisfactory; $F = 1$ if there is one element of unsatisfactory form: as one type of unsupported side; or two types of niche; or three types of directions when vertical and horizontal are counted as only one direction; or when there is no symmetry (V and R both 0); $F = 2$ if there are more than one of these elements of bad form.

C is the number of straight lines occupied by the sides of the polygon.

The ninety polygons listed below in order of decreasing M according to the empirical formula show the result of a rigid application of this formula.

It remains for the reader to determine for himself, by inspection of the polygons listed or by consideration of still further polygonal forms, whether or not the formula is to be regarded as reasonably satisfactory.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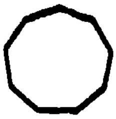

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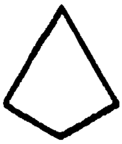

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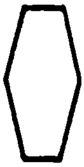

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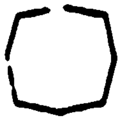

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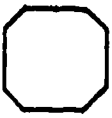

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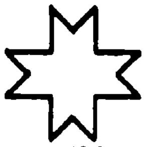

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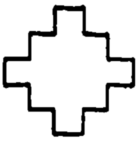

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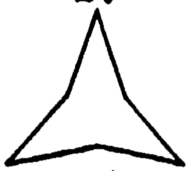

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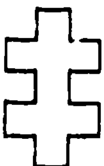

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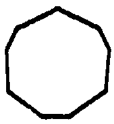

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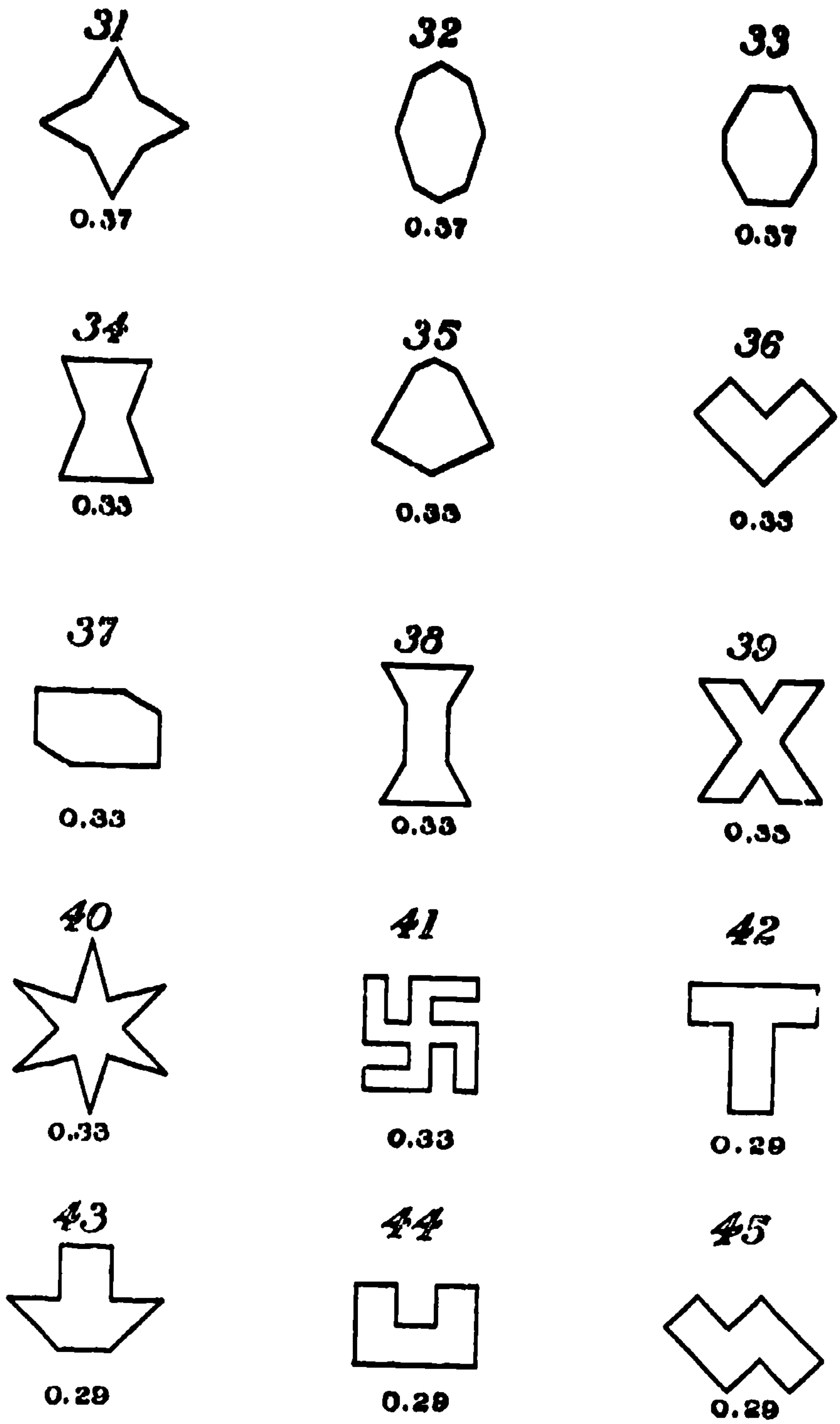
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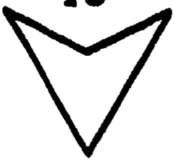

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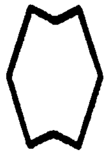

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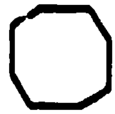

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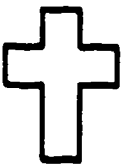

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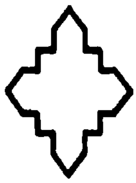

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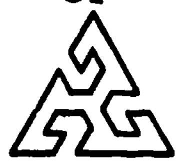

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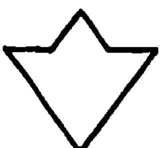

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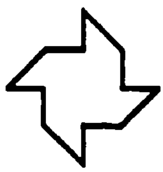

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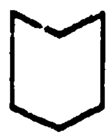

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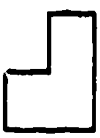

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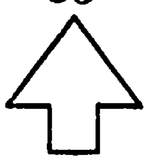

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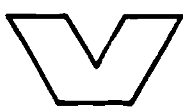

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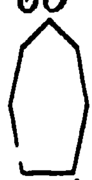

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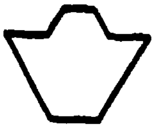

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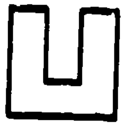

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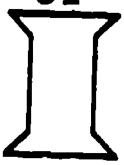

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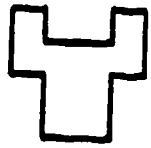

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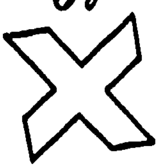

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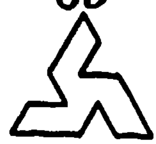

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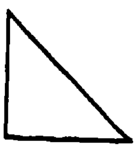

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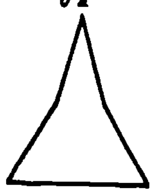

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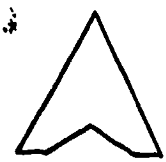

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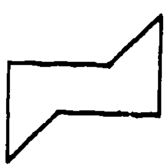

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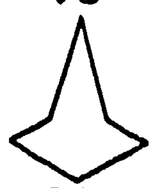

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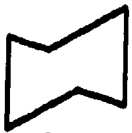

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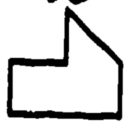

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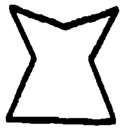

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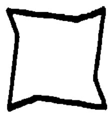

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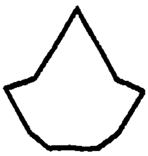

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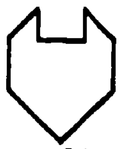

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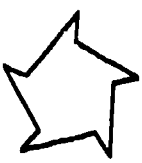

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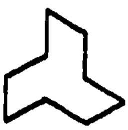

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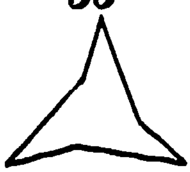

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